Modeling And Analysis Of An SVIRS Epidemic Model Involving External Sources Of Disease

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Abstract: In this paper a mathematical model that describes the flow of infectious disease in a population is proposed and studied. It is assumed that the disease divided the population into four classes: susceptible individuals (S), vaccinated individuals (V), infected individuals (I) and recover individuals (R). The impact of immigrants, vaccine and external sources of disease, on the dynamics of SVIRS epidemic model is investigated. The existence, uniqueness and boundedness of the solution of the model are discussed. The local and global stability of the model is studied. The occurrence of local bifurcation as well as Hopf bifurcation in the model is investigated. Finally the global dynamics of the proposed model is studied numerically.

Keywords: Epidemic models, Stability, Vaccinated, Immigrants, external sources, Local and Hopf bifurcation.

1. Introduction

Mathematical models have become important tools to study and analyze the spread and control of infectious disease. Most the proposed mathematical models those describe the transmission of infectious disease have been derived from the classical susceptible – infective – recover (SIR) model, which is suggested originally by Kermack and Mckenderick [1]. In that model the susceptible individuals become infective by contact with infected individuals and then the infected individuals may recover and transfer to removal individuals at a specific rate. Numbers of mathematical models were developed to study and analyze the spread of infectious diseases in order to prevent or minimize the transmission of them through quarantine and other measures see for example [2-5] and the references there in. On the other hand, since the resistance against an infectious disease represents protection that reduces an individual's risk of contracting the disease, therefore many epidemiological models involving vaccination (V) have been proposed and studied, see foe example [6-8] and the reference there in. Keeping the above in view, there are many infectious diseases spread within the population by direct contact between susceptible and infective individuals, they may spread through external sources in the environment such as (air, water, insects, etc…). Therefore, recently Das et al. [9] have been proposed and studied a mathematical model consisting of eco-epidemiological model involving external sources of disease. In this paper we proposed and studied a mathematical model consisting of SVIRS epidemic model involving immigrant individuals, some of them may arrive infected with the disease, and vaccine in which it is assumed that the disease transmitted by contact as well as external sources in the environment.

2. The mathematical model:

Consider a simple epidemiological model in which the total population (say N(t)) at time t is divided in to three sub classes the susceptible individuals S(t), infected individuals I(t) and recover individuals R(t). Such model can be represented as follows:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - \beta IS - \mu S \\ \frac{dI}{dt} &= \beta IS - (\mu + \alpha) I \\ \frac{dR}{dt} &= \alpha I - \mu R\end{aligned} \qquad (2.1)$$

Here $\Lambda > 0$ is the recruitment rate of the population, $\mu > 0$ is the natural death rate of the population, $\beta > 0$ is the infected rate (incidence rate) of the susceptible individuals due to direct contact with the infected individuals and $\alpha > 0$ is the natural recovery rate of the infected individuals. Now, since there are many infectious diseases (Alanfelonzha, bird's Anfelonzha and typhoid etc.) spread in the environment by different factors including insects, contact or other vectors, therefore, we assumed that the disease in the above model will transmitted between the population individuals by contact as well as external sources of disease in the environment with an external incidence rate $\beta_\circ \geq 0$. Also it is assumed that the lifetime of removal individual's immunity may not continue forever and then the removal individuals return to be susceptible class with a constant rate $\gamma \geq 0$ (also known as losing removal individual's immunity rate). Further, there is a constant flow, say $A > 0$, of a new members arriving into the population with the fraction p of A arriving infected ($0 \leq p \leq 1$). Then the above system (2.1) can be rewritten in the form:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda + (1-p)A - (\beta_\circ + \beta I)S - \mu S + \gamma R \\ \frac{dI}{dt} &= pA + (\beta_\circ + \beta I)S - (\mu + \alpha) I \\ \frac{dR}{dt} &= \alpha I - (\gamma + \mu) R\end{aligned} \qquad (2.2)$$

Keeping the above in view, in order to study the effect of vaccination on the system (2.2) let V(t) represented the vaccinated individuals in the population at time t , and then the following assumptions are made:

- The susceptible class is vaccinated at per capita rate $\psi \geq 0$.
- The infection can invade the susceptible class or vaccinated class depending on vaccine efficiency.
- The vaccine reduces the possibility of infection by a factor of σ, which is known as intensity vaccine immunity rate, where $0 \leq \sigma \leq 1$.
- The vaccine may not give a permanent immunity for susceptible individuals, so the vaccine may disappear and then the individuals loss the immunity with rate $0 \leq \theta \leq 1$.

Accordingly, the flow of disease in system (2.2) along with the above assumptions can be representing in the following block diagram:

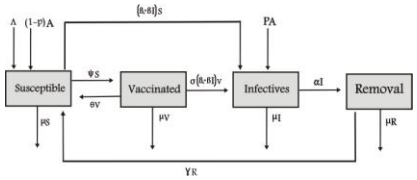


Fig. 1. Block diagram of system (2.3).

Therefore system (2) can be modified to:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda + (1-p)A - (\beta_\circ + \beta I)S - (\mu + \psi)S + \theta V + \gamma R \\
\frac{dV}{dt} &= \psi S - \sigma(\beta_\circ + \beta I)V - (\mu + \theta)V \\
\frac{dI}{dt} &= pA + (\beta_\circ + \beta I)S + \sigma(\beta_\circ + \beta I)V - (\mu + \alpha)I \\
\frac{dR}{dt} &= \alpha I - (\mu + \gamma)R
\end{aligned} \qquad (2.3)$$

Clearly for $\sigma = 0$ the vaccine is completely affective. While, $\sigma = 1$ stand for the situation where the vaccine is totally ineffective. On the other hand, $\theta = 0$ denotes to the case when immunity is life-long while $\theta = 1$ corresponds to the case where there is absolutely no vaccine induced immunity. Therefore at any point of time t the total number of population becomes

$$N = S(t) + V(t) + I(t) + R(t).$$

Obviously, due to the biological meaning of the variables S(t), V(t), I(t), and R(t), system (2.3) has the domain: $\Re_+^4 = \left\{(S,V,I,R) \in \Re^4, S \geq 0, V \geq 0, I \geq 0, R \geq 0\right\}$ which is positively invariant for system (2.3). Clearly, the interaction functions on the right hand side of system (2.3) are continuously differentiable. In fact they are Liptschizan function on $\Re_+^4$. Therefore the solution of system (2.3) exists and is unique. Further, all solutions of the system (2.3) with non-negative initial conditions are uniformly bounded as shown in the following theorem.

Theorem (2.1): All the solutions of system (2.3), which are initiate in $\Re_+^4$, are uniformly bounded.

Proof: Let (S(t), V(t), I(t), R(t)) be any solution of the system (2.3) with non-negative initial condition (S(0), V(0), I(0), R(0)), since N(t)= S(t)+V(t)+I(t)+R(t), then :

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dV}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$

Which gives

$$\frac{dN}{dt} + \mu N = \Lambda + A$$

Now, by solving the above linear differential equation, we get that the total population is asymptotically constant by:

$$N(t) = \frac{\Lambda + A}{\mu}$$

Hence all the solutions of system (2.3) that initiate in $\Re_+^4$, are confined in the region:

$$\zeta = \left\{(S,V,I,R) \in \Re_+^4 : N \leq \frac{\Lambda + A}{\mu} + \varepsilon; \varepsilon \geq 0\right\}$$

which is complete the proof.

3. Existence of Equilibrium points of system(2.3)

In this section, the existence of all possible equilibrium points of system (2.3) is discussed. Clearly, if $I = 0$, then the system (2.3) has an equilibrium point called a disease free equilibrium point and denoted by $E_\circ = (S_\circ, V_\circ, 0, 0)$ where:

$$\left.\begin{aligned} S_\circ &= \frac{(\Lambda + A)(\mu + \theta)}{\mu(\mu + \theta + \psi)} \\ V_\circ &= \frac{\psi(\Lambda + A)}{\mu(\mu + \theta + \psi)} \end{aligned}\right\} \qquad (3.1)$$

However, if $I \neq 0$ then the system (2.3) has an endemic equilibrium point denoted by $E_1 = (S_1, V_1, I_1, R_1)$ where S_1, V_1, I_1 and R_1 represent the positive solution of the following set of equations:

$$\left.\begin{aligned} &\Lambda + (1-p)A - (\beta_\circ + \beta I)S - (\mu + \psi)S + \theta V + \gamma R = 0 \\ &\psi S - \sigma(\beta_\circ + \beta I)V - (\mu + \theta)V = 0 \\ &pA + (\beta_\circ + \beta I)S + \sigma(\beta_\circ + \beta I)V - (\mu + \alpha)I = 0 \\ &\alpha I - (\mu + \gamma)R = 0 \end{aligned}\right. \qquad (3.2)$$

Straightforward computation to solve the above system of equations gives that:

$$\left.\begin{aligned} S_1 &= \frac{[\sigma(\beta_\circ+\beta I_1)+(\mu+\theta)]\{(\mu+\gamma)[\Lambda+(1-p)A]+\alpha\gamma I_1\}}{V} \\ V_1 &= \frac{\psi\{(\mu+\gamma)[\Lambda+(1-p)A]+\alpha\gamma I_1\}}{V} \\ R_1 &= \frac{\alpha I_1}{\mu+\gamma} \end{aligned}\right\}$$

$$V = (\beta_\circ+\beta I_1+\mu)\{(\mu+\gamma)[\sigma(\beta_\circ+\beta I_1)+\mu]+\theta(\mu+\gamma)\} + \psi(\mu+\gamma)[\sigma(\beta_\circ+\beta I_1)+\mu] \quad (3.3)$$

While I_1 is a positive root for the following third order equation:

$$D_1 I_1^3 + D_2 I_1^2 + D_3 I_1 + D_4 = 0 \quad (3.4)$$

here:

$$\begin{aligned} D_1 &= \sigma\mu\beta^2(\mu+\alpha+\gamma) < 0 \\ D_2 &= (\Lambda+A)[\sigma\beta^2(\mu+\gamma)] + \beta\alpha\gamma[2\sigma\beta_\circ+\sigma\psi+\mu+\theta] - (\mu+\alpha)(\mu+\gamma)\left[\theta+\beta(2\sigma\beta_\circ+\sigma\mu+\sigma\psi+\mu)\right] \\ D_3 &= (\mu+\theta)[pA\beta(\mu+\gamma)+\alpha\beta_\circ\gamma] + \sigma\alpha\gamma\beta_\circ(\beta_\circ+\psi)+(\mu+\gamma)\{\beta[\sigma pA\mu+(\Lambda+A)(2\sigma\beta_\circ+\sigma\psi+\mu+\theta)] \\ &\quad -[pA\beta(\mu+\theta)+(\mu+\alpha)[(\sigma\beta_\circ+\mu)(\beta_\circ+\mu+\psi)+\theta(\beta_\circ+\mu)]]\} \\ D_4 &= pA(\beta_\circ+\mu)(\mu+\gamma)\left[\sigma\beta_\circ+\mu+\theta\right] + pA\psi(\mu+\gamma)(\sigma\beta_\circ+\mu)+\beta_\circ(\mu+\gamma)\left[\Lambda+(1-p)A\right]\left[\sigma\beta_\circ+\sigma\psi+\mu+\theta\right] > 0 \end{aligned}$$

Clearly, equation (3.4) has a unique positive root given by I_1 and then E_1 exists uniquely in Int.$\Re_+^4$ if and only if at least one of the following two conditions hold.

$$(\Lambda+A)[\sigma\beta^2(\mu+\gamma)] + \beta\alpha\gamma[(2\sigma\beta_\circ+\sigma\psi+\mu+\theta] < (\mu+\alpha)(\mu+\gamma)\left[\theta+\beta(2\sigma\beta_\circ+\sigma\mu+\sigma\psi+\mu)\right] \quad (3.5a)$$

$$\beta[\sigma pA\mu+(\Lambda+A)(2\sigma\beta_\circ+\sigma\psi+\mu+\theta)] > pA\beta(\mu+\theta) + (\mu+\alpha)[(\sigma\beta_\circ+\mu)(\beta_\circ+\mu+\psi)+\theta(\beta_\circ+\mu)] \quad (3.5b)$$

4. Local stability analysis of system (2.3)

In this section, the local stability analysis of the equilibrium points $E_\circ$ and E_1 of the system (2.3) is studied as shown in the following theorems.

Theorem (4.1): The disease free equilibrium point $E_\circ = (S_\circ, V_\circ, 0, 0)$ of system (2.3) is locally asymptotically stable if the following sufficient condition is satisfied:

$$\beta(S_\circ+\sigma V_\circ) < \mu+\alpha \quad (4.1)$$

Proof: The Jacobian matrix of system (2.3) at $(E_\circ)$ can be written as:

$$J(E_\circ) = \begin{bmatrix} -(\mu+\psi) & \theta & -\beta S_\circ & \gamma \\ \psi & -(\mu+\theta) & -\sigma\beta V_\circ & 0 \\ 0 & 0 & \beta(S_\circ+\sigma V_\circ)-(\mu+\alpha) & 0 \\ 0 & 0 & \alpha & -(\mu+\gamma) \end{bmatrix} = [a_{ij}]_{4\times 4}$$

Clearly, $J(E_\circ)$ has the following eigenvalues:

$$\begin{aligned} \lambda_{S,V} &= -\frac{(2\mu+\psi+\theta)}{2} \pm \frac{1}{2}\sqrt{(2\mu+\psi+\theta)^2 - 4\mu(\mu+\psi+\theta)} \\ \lambda_I &= \beta(S_\circ+\sigma V_\circ)-(\mu+\alpha) \\ \lambda_R &= -(\mu+\gamma) \end{aligned}$$

here λ_k , $k = S, V, I, R$ represents the eigenvalue in k-direction. Obviously, λ_S and λ_V have negative real parts, while $\lambda_R < 0$. Therefore $E_\circ$ is locally asymptotically stable if and only if the eigenvalue $\lambda_I < 0$, which is satisfied provided that condition (4.1) holds and hence the proof is complete. ■

Theorem (4.2): Assume that, the endemic equilibrium point $E_1 = (S_1, V_1, I_1, R_1)$ of system (2.3) exists in the Int.$\Re_+^4$. Then it is locally asymptotically stable if the following condition is satisfied:

$$\mu > 2\beta(S_1+\sigma V_1) \quad (4.2)$$

Proof: The Jacobian matrix of system (2.3) at the endemic equilibrium point E_1 that denoted by $J(E_1)$ can be written:

$$\begin{bmatrix} -(\beta_\circ+\beta I_1)-(\mu+\psi) & \theta & -\beta S_1 & \gamma \\ \psi & -(\sigma\beta_\circ+\sigma\beta I_1+\mu+\theta) & -\sigma\beta V_1 & 0 \\ \beta_\circ+\beta I_1 & \sigma(\beta_\circ+\beta I_1) & \beta(S_1+\sigma V_1)-(\mu+\alpha) & 0 \\ 0 & 0 & \alpha & -(\mu+\gamma) \end{bmatrix} = [b_{ij}]_{4\times 4}$$

Now, according to Gersgorin theorem if the following condition holds:

$$|b_{ii}| > \sum_{\substack{i=1 \\ i\neq j}}^{4} |b_{ij}|$$

Then all eigenvalues of $J(E_1)$ exists in the region:

$$\wp = \bigcup\left\{ U^* \in C : \left|U^* - b_{ii}\right| < \sum_{\substack{i=1 \\ i\neq j}}^{4} |b_{ij}| \right\}$$

Therefore, according to the given condition (4.2) all the eigenvalues of $J(E_1)$ exists in the left half plane and hence, E_1 is locally asymptotically stable.

5. Global stability analysis of system (2.3)

In this section, the global dynamics of system (2.3) is studied with the help of Lyapunov function as shown in the following theorems.

Theorem (5.1): Assume that, the disease free equilibrium point $E_\circ$ of system (2.3) is locally asymptotically stable. Then the basin of attraction of $E_\circ$, say $B(E_\circ) \subset \Re_+^4$, it is globally asymptotically stable if satisfy the following conditions:

$$\left(\frac{\theta}{S}+\frac{\psi}{V}\right)^2 < 4\left(\frac{\mu+\psi}{S}\right)\left(\frac{\theta+\mu}{V}\right) \qquad (5.1a)$$

$$\frac{pAS_\circ}{S}+(\beta_\circ+\beta I)(S_\circ+\sigma V_\circ) < \left(\frac{\gamma S_\circ}{S}+\mu\right)R+\mu I \quad (5.1b)$$

Proof: Consider the following positive definite function:

$$W_1 = \left(S-S_\circ-S_\circ \ln\frac{S}{S_\circ}\right)+\left(V-V_\circ-V_\circ \ln\frac{V}{V_\circ}\right)+I+R$$

Clearly, $W_1 : R_+^4 \to R$ is a continuously differentiable function such that $W_1(S_\circ,V_\circ,0,0)=0$, and $W_1(S,V,I,R)>0,\ \forall (S,V,I,R)\neq(S_\circ,V_\circ,0,0)$. Further we have:

$$\frac{dW_1}{dt} = \left(\frac{S-S_\circ}{S}\right)\frac{dS}{dt}+\left(\frac{V-V_\circ}{V}\right)\frac{dV}{dt}+\frac{dI}{dt}+\frac{dR}{dt}$$

By simplifying this equation we get:

$$\begin{aligned}\frac{dW_1}{dt} = &-\frac{(\mu+\psi)}{S}(S-S_\circ)^2+\left(\frac{\theta}{S}+\frac{\psi}{V}\right)(S-S_\circ)(V-V_\circ)\\ &-\frac{(\mu+\theta)}{V}(V-V_\circ)^2+\frac{pAS_\circ}{S}+(\beta_\circ+\beta I)(S_\circ+\sigma V_\circ)\\ &-\left[\left(\frac{\gamma S_\circ}{S}+\mu\right)R+\mu I\right]\end{aligned}$$

Therefore, according to condition (5.1a) it is obtain that:

$$\begin{aligned}\frac{dW_1}{dt} \le &-\left[\sqrt{\frac{\mu+\psi}{S}}(S-S_\circ)-\sqrt{\frac{\mu+\theta}{V}}(V-V_\circ)\right]^2+\frac{pAS_\circ}{S}\\ &+(\beta_\circ+\beta I)(S_\circ+\sigma V_\circ)-\left[\left(\frac{\gamma S_\circ}{S}+\mu\right)R+\mu I\right]\end{aligned}$$

Obviously $\frac{dW_1}{dt}<0$ for every initial points satisfying condition (5.1b) and then W_1 is a Lyapunov function provided that conditions (5.1a)-(5.1b) hold. Thus $E_\circ$ is globally asymptotically stable in the interior of $B(E_\circ)$, which means that $B(E_\circ)$ is the basin of attraction and that complete the proof.

Theorem (5.2): Let the endemic equilibrium point E_1 of system (2.3) is locally asymptotically stable. Then it is globally asymptotically stable provided that:

$$\beta(S_1+\sigma V_1) < \mu+\alpha \qquad (5.2a)$$

$$\left[\beta(S_1+I_1)+\beta_\circ\right]^2 < \frac{4}{9}\left[\beta_\circ+\beta I+\mu+\psi\right]\left[\mu+\alpha-\beta(S_1+\sigma V_1)\right] \qquad (5.2b)$$

$$\left[\theta+\psi\right]^2 < \frac{2}{3}\left[\beta_\circ+\beta I+\mu+\psi\right]\left[\mu+\theta+\sigma(\beta_\circ+\beta I)\right] \qquad (5.2c)$$

$$\gamma^2 < \frac{2}{3}\left[\beta_\circ+\beta I+\mu+\psi\right]\left[\mu+\gamma\right] \qquad (5.2d)$$

$$\left[\sigma(\beta_\circ+\beta(I_1-V_1))\right]^2 < \frac{2}{3}\left[\mu+\theta+\sigma(\beta_\circ+\beta I)\right]\left[\mu+\alpha-\beta(S_1+\sigma V_1)\right] \qquad (5.2e)$$

$$\alpha^2 < \frac{2}{3}\left[\mu+\alpha-\beta(S_1+\sigma V_1)\right]\left[\mu+\gamma\right] \qquad (5.2h)$$

Proof: Consider the following positive definite function:

$$W_2 = \frac{(S-S_1)^2}{2}+\frac{(V-V_1)^2}{2}+\frac{(I-I_1)^2}{2}+\frac{(R-R_1)^2}{2}$$

Clearly, $W_2 : R_+^4 \to R$ is a continuously differentiable function such that $W_2(S_1,V_1,I_1,R_1)=0$ and $W_2(S,V,I,R)>0,\ \forall (S,V,I,R)\neq(S_1,V_1,I_1,R_1)$ Further, we have:

$$\frac{dW_2}{dt} = (S-S_1)\frac{dS}{dt}+(V-V_1)\frac{dV}{dt}+(I-I_1)\frac{dI}{dt}+(R-R_1)\frac{dR}{dt}$$

By simplifying this equation we get:

$$\begin{aligned}\frac{dW_2}{dt} = &-\frac{1}{3}q_{11}(S-S_1)^2-\frac{1}{3}q_{33}(I-I_1)^2\\ &+q_{13}(S-S_1)(I-I_1)-\frac{1}{3}q_{11}(S-S_1)^2\\ &-\frac{1}{2}q_{22}(V-V_1)^2+q_{12}(S-S_1)(V-V_1)\\ &-\frac{1}{3}q_{11}(S-S_1)^2-\frac{1}{2}q_{44}(R-R_1)^2\\ &+q_{14}(S-S_1)(R-R_1)-\frac{1}{2}q_{22}(V-V_1)^2\\ &-\frac{1}{3}q_{33}(I-I_1)^2+q_{23}(V-V_1)(I-I_1)-\frac{1}{3}q_{33}(I-I_1)^2\\ &-\frac{1}{2}q_{44}(R-R_1)^2+q_{34}(I-I_1)(R-R_1)\end{aligned}$$

With

$$\begin{aligned}&q_{11}=\beta_\circ+\beta I+\mu+\psi, q_{13}=\beta S_1+\beta_\circ+\beta I_1,\\ &q_{33}=\mu+\alpha-\beta(S_1+\sigma V_1), q_{12}=\theta+\psi,\\ &q_{22}=\mu+\theta+\sigma(\beta_\circ+\beta I), q_{44}=\mu+\gamma, q_{14}=\gamma,\\ &q_{23}=\sigma(\beta_\circ+\beta I_1-\beta V_1), q_{34}=\alpha\end{aligned}$$

Therefore, according to the conditions (5.2a)-(5.2h) we obtain that:

$$\frac{dW_2}{dt} \le -\left[\sqrt{\frac{q_{11}}{3}}(S-S_1)-\sqrt{\frac{q_{33}}{3}}(I-I_1)\right]^2 -\left[\sqrt{\frac{q_{11}}{3}}(S-S_1)-\sqrt{\frac{q_{22}}{2}}(V-V_1)\right]^2 -\left[\sqrt{\frac{q_{11}}{3}}(S-S_1)-\sqrt{\frac{q_{44}}{2}}(R-R_1)\right]^2 -\left[\sqrt{\frac{q_{22}}{2}}(V-V_1)-\sqrt{\frac{q_{33}}{3}}(I-I_1)\right]^2 -\left[\sqrt{\frac{q_{33}}{3}}(I-I_1)-\sqrt{\frac{q_{44}}{2}}(R-R_1)\right]^2$$

Clearly, $\frac{dW_2}{dt} < 0$, and then W_2 is a Lyapunov function provided that the given conditions hold. Therefore, E_1 is globally asymptotically stable. ■

6. The local bifurcation analysis of system (2.3)

In this section, the occurrence of local bifurcations (such as saddle-node, transcritical and pitchfork) near the equilibrium points of system (2.3) is studied in the following theorem.

Theorem (6.1): System (2.3) has a transcritical bifurcation near the disease free equilibrium point $E_\circ$, but neither saddle-node bifurcation, nor pitchfork bifurcation can accrue at the parameter

$$\mu_\circ = \beta(S_\circ + \sigma V_\circ) - \alpha \qquad (6.1)$$

Proof: It is easy to verify that the Jacobian matrix of system (2.3) at $(E_\circ, \mu_\circ)$ can be written as:

$$J_{\mu_\circ} = Df(E_\circ, \mu_\circ) = \begin{bmatrix} -(\mu_\circ+\psi) & \theta & -\beta S_\circ & \gamma \\ \psi & -(\mu_\circ+\theta) & -\sigma\beta V_\circ & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & -(\mu_\circ+\gamma) \end{bmatrix}$$

here $a_{11}(\mu_\circ) = -(\mu_\circ+\psi),\ a_{22}(\mu_\circ) = -(\mu_\circ+\theta),\ a_{44}(\mu_\circ) = -(\mu_\circ+\gamma)$. Clearly, the third eigenvalue λ_I in I-direction is zero $(\lambda_I = 0)$, further the eigenvector (say $K = (k_1, k_2, k_3, k_4)^T$) corresponding to $\lambda_I = 0$ satisfy the following:

$J_{\mu_\circ} K = \lambda K$ then $J_{\mu_\circ} K = 0$

From which we get that:

$$-(\mu_\circ+\psi)k_1 + \theta k_2 - \beta S_\circ k_3 + \gamma k_4 = 0 \qquad (6.2a)$$

$$\psi k_1 - (\mu_\circ+\theta)k_2 - \sigma\beta V_\circ k_3 = 0 \qquad (6.2b)$$

$$\alpha k_3 - (\mu_\circ+\gamma)k_4 = 0 \qquad (6.2c)$$

So by solving the above system of equations we get:

$$k_1 = -xk_3;\ k_2 = -yk_3;\ k_4 = zk_3$$

Where:

$$x = \frac{\{\sigma\beta\theta(\mu_\circ+\gamma)V_\circ + (\mu_\circ+\theta)(\beta S_\circ(\mu_\circ+\gamma)+\alpha\gamma)\}}{2\mu_\circ}$$

$$y = \frac{\{\psi[\sigma\beta\theta(\mu_\circ+\gamma)V_\circ + (\mu_\circ+\theta)(\beta S_\circ(\mu_\circ+\gamma)+\alpha\gamma)] + 2\sigma\beta\mu_\circ V_\circ\}}{2\mu_\circ(\mu_\circ+\theta)}$$

$$z = \frac{\alpha}{(\mu_\circ+\gamma)}$$

Here k_3 be any non zero real number. Thus

$$K = \begin{bmatrix} -xk_3 \\ -yk_3 \\ k_3 \\ zk_3 \end{bmatrix}$$

Similarly the eigenvector $W = (w_1, w_2, w_3, w_4)^T$ that corresponding to $\lambda_I = 0$ of $J^T_{\mu_\circ}$ can be written:

$$J^T_{\mu_\circ} \cdot W = \begin{bmatrix} -(\mu_\circ+\psi) & \psi & 0 & 0 \\ \theta & -(\mu_\circ+\theta) & 0 & 0 \\ -\beta S_\circ & -\sigma\beta V_\circ & 0 & \alpha \\ \gamma & 0 & 0 & -(\mu_\circ+\gamma) \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = 0$$

This gives:

$$W = \begin{bmatrix} 0 \\ 0 \\ w_3 \\ 0 \end{bmatrix}$$

Here w_3 is any non-zero real number. Now rewrite system (2.3) in a vector form as:

$$\frac{dX}{dt} = f(X)$$

Where $X = (S, V, I, R)^T$ and $f = (f_1, f_2, f_3, f_4)^T$ with $f_i, i = 1,2,3,4$ are given in system (2.3), and then determine $\frac{df}{d\mu} = f_\mu$ we get that:

$$f_\mu = \begin{bmatrix} -S \\ -V \\ -I \\ -R \end{bmatrix} \quad \text{then} \quad f_\mu(E_\circ, \mu_\circ) = \begin{bmatrix} -S_\circ \\ -V_\circ \\ 0 \\ 0 \end{bmatrix}$$

Therefore:

$$W^T \cdot f_\mu(E_\circ, \mu_\circ) = 0$$

Consequently, according to Sotomayor theorem [10] the system (2.3) has no saddle-node bifurcation near $E_\circ$ at $\mu_\circ$. Now in order to investigate the accruing of other types of bifurcation, the derivative of f_μ with respect to vector X, say $Df_\mu(E_\circ,\mu_\circ)$, is computed

$$Df_\mu(E_\circ,\mu_\circ)=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

So

$$W^T\cdot\left[Df_\mu(E_\circ,\mu_\circ)\cdot K\right]=-k_3w_3\neq 0$$

Again, according to Sotomayor theorem, if in addition to the above, the following holds

$$W^T\cdot\left[D^2f(E_\circ,\mu_\circ)\cdot(K,K)\right]\neq 0$$

here $Df(E_\circ,\mu_\circ)$ is the Jacobian matrix at $E_\circ$ and $\mu_\circ$, then the system (2.3) possesses a transcritical bifurcation but no pitch-fork bifurcation can occur. Now since we have that:

$$\left[D^2f(E_\circ,\mu_\circ)\cdot(K,K)\right]=\begin{bmatrix}2x\beta k_3^2\\2y\beta k_3^2\\-\beta(x+\sigma y)(1+k_3)k_3\\0\end{bmatrix}$$

Therefore:

$$W^T\cdot\left[D^2f(E_\circ,\mu_\circ)\cdot(K,K)\right]=-\beta(x+\sigma y)(1+k_3)k_3w_3\neq 0$$

Then the system (2.3) has a transcritical bifurcation at $E_\circ$ when the parameter μ passes through the bifurcation value $\mu_\circ$. ■

7. SIRS epidemic model without vaccination

Consider the proposed system (2.3) in case of absence of vaccine, that is when the parameter $\psi=0$, then system (2.3) will be reduced to the following subsystem.

$$\begin{aligned}\frac{dS}{dt}&=\Lambda+(1-p)A-(\beta_\circ+\beta I)S-\mu S+\gamma R=g_1(S,I,R)\\\frac{dI}{dt}&=pA+(\beta_\circ+\beta I)S-(\mu+\alpha)I=g_2(S,I,R)\\\frac{dR}{dt}&=\alpha I-(\mu+\gamma)R=g_3(S,I,R)\end{aligned}\qquad(7.1)$$

Clearly, system (7.1) represents a simple SIRS epidemic model involving disease, transmitted by contact and by external sources in the environment, with partially infective immigrant's individuals. Obviously the above subsystem is uniformly bounded and has two non negative equilibrium points: First the disease free equilibrium point that denoted by $E_2=(S_2,0,0)$, which always exists where:

$$S_2=\frac{\Lambda+A}{\mu}\qquad(7.2)$$

Second the endemic equilibrium point of system (7.1) that denoted by $E_3=(S_3,I_3,R_3)$, which exists uniquely in the Int. $\Re_+^3=\left\{(S,I,R)\in\Re_+^3,S\geq 0,I\geq 0,R\geq 0\right\}$, where:

$$S_3=\frac{(\mu+\gamma)\left[\Lambda+(1-p)A\right]+\alpha\gamma I_3}{(\mu_\circ+\gamma)(\beta_\circ+\beta I_3+\mu)}\qquad(7.3a)$$

$$R_3=\frac{\alpha I_3}{\mu+\gamma}\qquad(7.3b)$$

$$I_3=\frac{-D_2}{2D_1}-\frac{1}{2D_1}\sqrt{D_2^2-4D_1D_3}\qquad(7.3c)$$

here:

$$\begin{aligned}D_1&=-\beta\mu(\mu+\alpha+\gamma)<0\\D_2&=\beta(\mu+\gamma)-\mu\left[(\mu^2+\mu\beta_\circ+\mu\gamma+\gamma\beta_\circ)+\alpha(\mu+\beta_\circ+\gamma)\right]\\D_3&=(\mu+\gamma)\left[pA(\beta_\circ+\mu)+\beta_\circ(\Lambda+(1-p)A\right]>0\end{aligned}$$

The local stability analysis of the above equilibrium points E_2 and E_3 of system (7.1) is studied as shown in the following theorems.

Theorem (7.1): The disease free equilibrium point $E_2=(S_2,0,0)$ of system (7.1) is locally asymptotically stable provided that:

$$\beta S_2<\mu+\alpha\qquad(7.4a)$$

While it is a saddle point provided that:

$$\beta S_2>\mu+\alpha\qquad(7.4b)$$

Proof: The Jacobian matrix of system (7.1) at E_2 can be written as:

$$J(E_2)=\begin{bmatrix}-\mu&-\beta S_2&\gamma\\0&\beta S_2-(\mu+\alpha)&0\\0&\alpha&-(\mu+\gamma)\end{bmatrix}$$

Clearly, $J(E_2)$ has the following eigenvalues:

$$\begin{aligned}&\lambda_S=-\mu<0\ ;\ \lambda_I=\beta S_2-(\mu+\alpha)\ ;\\&\lambda_R=-(\mu+\gamma)<0\end{aligned}\qquad(7.5)$$

here λ_k , $k = S, I, R$ represents the eigenvalue in k-direction. Obviously, λ_S and λ_R are negative and then E_2 is locally asymptotically stable if and only if the eigenvalue $\lambda_I < 0$, which is satisfied provided that condition (7.4a) holds, while it is saddle point provided that condition (7.4b) holds and hence the proof is complete.

Theorem (7.2): The endemic equilibrium point $E_3 = (S_3, I_3, R_3)$ of system (7.1) is locally asymptotically stable provided that:

$$\beta S_3 < \mu + \alpha \qquad (7.6a)$$

$$\alpha\gamma < \beta S_3(\mu + \gamma) \qquad (7.6b)$$

Proof: The Jacobian matrix of system (7.1) at the endemic equilibrium point E_3 can be written:

$$J(E_3) = \begin{bmatrix} -(\beta_\circ + \beta I_3 + \mu) & -\beta S_3 & \gamma \\ \beta_\circ + \beta I_3 & \beta S_3 - (\mu + \alpha) & 0 \\ 0 & \alpha & -(\mu + \gamma) \end{bmatrix} = \left[d_{ij}\right]_{3\times 3}$$

Then the characteristic equation of $J(E_3)$ is given by:

$$\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 = 0 \qquad (7.7)$$

here:

$$\begin{aligned}
\Omega_1 &= -\left[d_{11} + d_{22} + d_{33}\right] \\
&= (\beta_\circ + \beta I_3 + \mu) - (\beta S_3 - (\mu + \alpha)) + (\mu + \gamma) \\
\Omega_2 &= d_{11}d_{22} - d_{12}d_{21} + d_{11}d_{33} + d_{22}d_{33} \\
\Omega_3 &= -\left[d_{11}d_{22}d_{33} + d_{21}(d_{13}d_{32} - d_{12}d_{33})\right] \\
&= (\beta_\circ + \beta I_3 + \mu)\{(\mu + \gamma)[(\mu + \alpha) - \beta S_3]\} \\
&\qquad - (\beta_\circ + \beta I_3)(\alpha\gamma - \beta S_3(\mu + \gamma))
\end{aligned}$$

Further:

$$\begin{aligned}
\Delta &= \Omega_1\Omega_2 - \Omega_3 \\
&= -d_{11}d_{22}(d_{11} + d_{22}) - d_{11}d_{33}(d_{11} + d_{33}) \\
&\quad - d_{22}d_{33}(d_{22} + d_{33}) + d_{12}d_{21}(d_{11} + d_{22}) \\
&\qquad - 2d_{11}d_{22}d_{33} \\
&= \left[(\beta_\circ + \beta I_3 + \mu)((\mu + \alpha) - \beta S_3)\right] \\
&\qquad \left[(\beta_\circ + \beta I_3 + \mu) + ((\mu + \alpha) - \beta S_3\right] \\
&\quad + \left[(\beta_\circ + \beta I_3 + \mu)(\mu + \gamma)\right]\left[(\beta_\circ + \beta I_3 + \mu) + (\mu + \gamma)\right] \\
&\quad + \left[((\mu + \alpha) - \beta S_3)(\mu + \gamma)\right]\left[((\mu + \alpha) - \beta S_3) + (\mu + \gamma)\right] \\
&\quad + \left[(\beta S_3)(\beta_\circ + \beta I_3)\right]\left[(\beta_\circ + \beta I_3 + \mu)((\mu + \alpha) - \beta S_3)\right] \\
&\qquad + 2(\beta_\circ + \beta I_3 + \mu)((\mu + \alpha) - \beta S_3)(\mu + \gamma)
\end{aligned}$$

Now according to (Routh-Hurtiz) criterion E_3 will be locally asymptotically stable provided that $\Omega_1 > 0$; $\Omega_3 > 0$ and $\Delta = \Omega_1\Omega_2 - \Omega_3 > 0$. Clearly: $\Omega_1 > 0$; $\Omega_3 > 0$ and $\Delta = \Omega_1\Omega_2 - \Omega_3 > 0$ provided that conditions (7.6a)-(7.6b) are hold. Hence the proof is complete. The global dynamics of system (7.1) is studied with the help of Lyapunov function as shown in the following theorems.

Theorem (7.3): Assume that, the disease free equilibrium point E_2 of system (7.1) is locally asymptotically stable. Then the basin of attraction of E_2, say $B(E_2) \subset \Re^3_+$, satisfy the following condition:

$$pA + (\beta_\circ + \beta I)S < \gamma R \qquad (7.8)$$

Proof: Consider the following positive definite function:

$$L_1 = \int_{S_2}^{S} \left(\frac{F - S_2}{F}\right)\frac{dF}{dt} + I + R$$

Clearly, $L_1 : R^3_+ \to R$ is a continuously differentiable function such that $L_1(S_2, 0, 0) = 0$, and $L_1(S, I, R) > 0$, $\forall (S, I, R) \neq (S_2, 0, 0)$.

Further, we have:

$$\frac{dL_1}{dt} = \left(\frac{S - S_2}{S}\right)\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$

By simplifying this equation we get:

$$\frac{dL_1}{dt} = -\frac{\mu}{S}(S - S_2)^2 - \mu(I + R) + \left[\frac{pA}{S} + (\beta_\circ + \beta I) - \frac{\gamma R}{S}\right]S_2$$

Obviously, $\frac{dL_1}{dt} < 0$ for every initial point satisfying condition (7.8) and then L_1 is a Lyapunov function provided that condition (7.8) holds. Thus E_2 is globally asymptotically stable in the interior of $B(E_2)$, which means that $B(E_2)$ is the basin of attraction and this complete the proof.

Theorem (7.4): Let the endemic equilibrium point E_3 of system (7.1) is locally asymptotically stable. Then it is globally asymptotically stable provided that the following conditions are satisfied:

$$(\beta_\circ + \beta I)^2 < (\beta_\circ + \beta I + \mu)(\mu + \alpha) \qquad (7.9a)$$

$$\gamma^2 < (\beta_\circ + \beta I + \mu)(\mu + \gamma) \qquad (7.9b)$$

$$\alpha^2 < (\mu + \alpha)(\mu + \gamma) \qquad (7.9c)$$

Proof: Consider the following positive definite function:

$$L_2 = \frac{(S - S_3)^2}{2} + \frac{(I - I_3)^2}{2} + \frac{(R - R_3)^2}{2}$$

Clearly, $L_2 : R_+^3 \to R$ is a continuously differentiable function such that $L_2(S_3, I_3, R_3) = 0$ and $L_2(S, I, R) > 0$, $\forall (S, I, R) \neq (S_3, I_3, R_3)$. Further, we have:

$$\frac{dL_2}{dt} = (S - S_3)\frac{dS}{dt} + (I - I_3)\frac{dI}{dt} + (R - R_3)\frac{dR}{dt}$$

By simplifying this equation we get:

$$\begin{aligned}\frac{dL_2}{dt} = &-\frac{1}{2}q_{11}(S - S_3)^2 - \frac{1}{2}q_{22}(I - I_3)^2 + q_{12}(S - S_3)(I - I_3) \\ &-\frac{1}{2}q_{11}(S - S_3)^2 - \frac{1}{2}q_{33}(R - R_3)^2 + q_{13}(S - S_3)(R - R_3) \\ &-\frac{1}{2}q_{22}(I - I_3)^2 - \frac{1}{2}q_{33}(R - R_3)^2 + q_{23}(I - I_3)(R - R_3)\end{aligned}$$

with

$$q_{11} = \beta_\circ + \beta I + \mu \ ; \ q_{12} = \beta_\circ + \beta I \ ;$$
$$q_{22} = \mu + \alpha \ ; q_{13} = \gamma \ ; q_{33} = \mu + \gamma \ ; \ q_{23} = \alpha$$

Therefore, according to the conditions (7.9a)-(7.9c), we obtain that:

$$\begin{aligned}\frac{dL_2}{dt} \leq &-\left[\sqrt{\frac{q_{11}}{2}}(S - S_3) - \sqrt{\frac{q_{22}}{2}}(I - I_3)\right]^2 \\ &-\left[\sqrt{\frac{q_{11}}{2}}(S - S_3) - \sqrt{\frac{q_{33}}{2}}(R - R_3)\right]^2 \\ &-\left[\sqrt{\frac{q_{22}}{2}}(I - I_3) - \sqrt{\frac{q_{33}}{2}}(R - R_3)\right]^2\end{aligned}$$

Clearly, $\frac{dL_2}{dt} < 0$, and then L_2 is a Lyapunov function provided that the given conditions (7.9a)-(7.9c) hold. Therefore, E_3 is globally asymptotically stable and hence the proof is complete. The occurrence of local bifurcations (such as saddle-nod, transcritical and pitchfork) near the equilibrium point of system (7.1) is studied in the following theorem.

Theorem (7.5): System (7.1) has a transcritical bifurcation near the disease free equilibrium point E_2, but neither saddle-node bifurcation, nor pitchfork bifurcation can accrue at the parameter

$$\beta^* = \frac{\mu + \alpha}{S_2} \qquad (7.10)$$

Proof: It is easy to check that the Jacobian matrix of system (7.1) at (E_2, β^*) can be written as:

$$J = J(E_2, \beta^*) = \begin{bmatrix} -\mu & -\beta^* S_2 & \gamma \\ 0 & 0 & 0 \\ 0 & \alpha & -(\mu + \gamma) \end{bmatrix}$$

Clearly, the second eigenvalue λ_I in I-direction is zero $(\lambda_I = 0)$, while λ_S and λ_R those are given in equation (7.7) are negative. Further, the eigenvector (say $K = (k_1, k_2, k_3)^T$) corresponding to $\lambda_I = 0$ can be written as:

$$K = \begin{bmatrix} -zk_2 \\ k_2 \\ yk_2 \end{bmatrix}$$

here $k_1 = -zk_2$; $k_3 = yk_2$, k_2 be any non zero real number with $z = \frac{\mu + \alpha + \gamma}{\mu + \gamma}$ and $y = \frac{\alpha}{\mu + \gamma}$. Similarly the eigenvector $W = (w_1, w_2, w_3)^T$ corresponding to $\lambda_I = 0$ of J^T can be written:

$$W = \begin{bmatrix} 0 \\ w_2 \\ 0 \end{bmatrix}$$

Here w_2 is any non-zero real number. Now rewrite system (7.1) in a vector form as:

$$\frac{dX}{dt} = g(X)$$

Where $X = (S, I, R)^T$ and $g = (g_1, g_2, g_3)^T$ with g_i , $i = 1,2,3$ are given in system (7.1), and then determine $\frac{dg}{d\beta} = g_\beta$ we get that:

$$W^T \cdot g_\beta(E_2, \beta^*) = 0$$

Consequently, according to Sotomayor theorem [10] the system has no saddle-node bifurcation near E_2 at β^*. Now in order to investigate the accruing of other types of bifurcation, the derivative of g_β with respect to vector X, say $Dg_\beta(E_2, \beta^*)$, is computed and then we obtain that:

$$W^T \cdot \left[Dg_\beta(E_2, \beta^*) \cdot K\right] = S_2 k_2 w_2 \neq 0$$

Moreover, since

$$W^T \cdot \left[D^2 g(E_2, \beta^*) \cdot (K, K)\right] = -2\beta z w_2 k_2^2 \neq 0$$

Then the system (7.1) has a trnscritical bifurcation but not pitch-fork bifurcation at E_2 when the parameter β passes through the bifurcation value β^*. The occurrence of Hopf-bifurcation near the endemic equilibrium point of system (7.1) is also studied. Not that, it is well known that the necessary conditions of the three dimensional dynamical system (7.1) to have a Hopf bifurcation around E_3 at a specific parameter value (say q^*) are given by $\Omega_1(q^*) > 0$, $\Omega_2(q^*) > 0$ and $\Delta(q^*) = \Omega_1(q^*)\Omega_2(q^*) - \Omega_3(q^*) = 0$, where Ω_1 and Ω_2 represent the coefficients of the characteristic equation of the dynamical system (7.1). Now since the conditions that guarantee the positivity of Ω_1 and Ω_2 are the same conditions that guarantee the positivity of $\Delta = \Omega_1\Omega_2 - \Omega_3$. Hence there is no possibility of occurrence of Hopf bifurcation.

8. Numerical analysis of systems (2.3) and (7.1):

In this section, the global dynamics of systems (2.3) and (7.1) is studied numerically. The objectives of this study are confirming our obtained analytical results and understand the effects of immigration, existence of vaccine and existence of the external sources for disease on the dynamics of SVIRS and SIRS epidemic models. Consequently, first system (2.3) is solved numerically for different sets of initial conditions and for different sets of parameters. It is observed that, for the following set of hypothetical parameters that satisfies stability condition (4.1) of disease free equilibrium point, system (2.3) has a globally asymptotically stable disease free equilibrium point as shown in following figure.

$$\Lambda = 400,\ A = 100,\ p = 0,\ \beta = 0.0005,\ \beta_\circ = 0,$$
$$\mu = 0.1, \psi = 0.5,\ \theta = 0.05,\ \sigma = 0.01, \qquad (7.11)$$
$$\alpha = 0.8,\ \gamma = 0.5$$

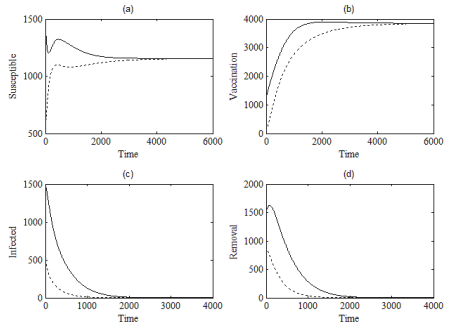


Fig. 2, Time series of the solution of system (2.3). (a) trajectories of S, (b) trajectories of V, (c) trajectories of I and (d) trajectories of R. The solid line refers to the trajectory started at (1500,1200,1500,1500) while dotted line refers to trajectory started at (500,400,500,900).

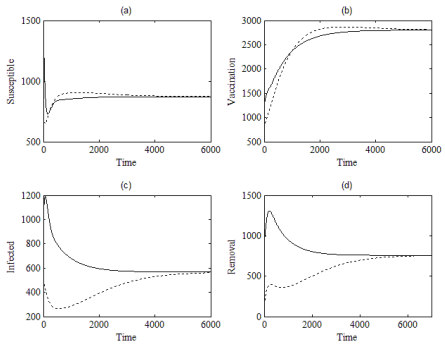


Fig. 3, Time series of the solution of system (2.3). (a) trajectories of S , (b) trajectories of V, (c) trajectories of I and (d) trajectories of R. the solid line refers to the trajectory started at (1500,1200,1000,900) while the dotted line refers to the trajectory started at (700,800,500,100).

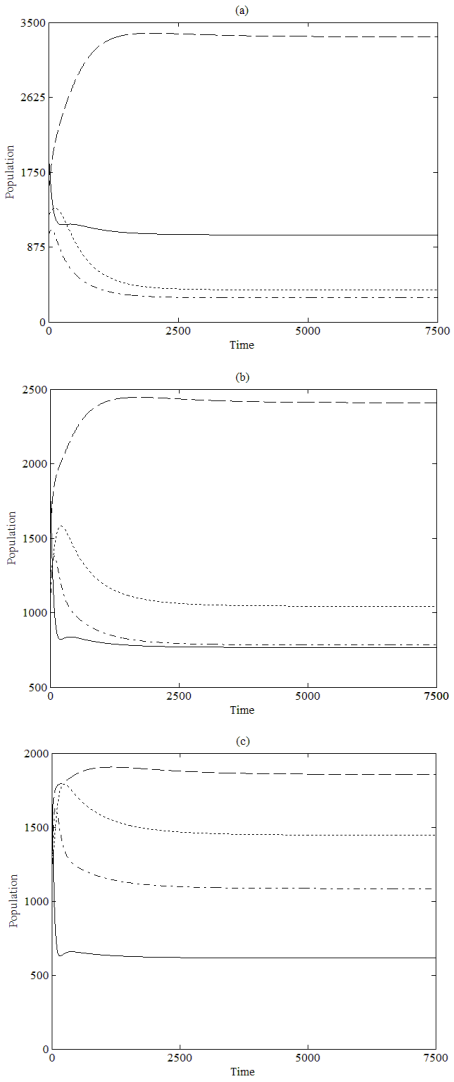


Fig. 4, Time series of the solution of system (2.3). (a) for $\beta_0 = 0.1$, (b) for $\beta_0 = 0.5$, (c) for $\beta_0 = 1$.

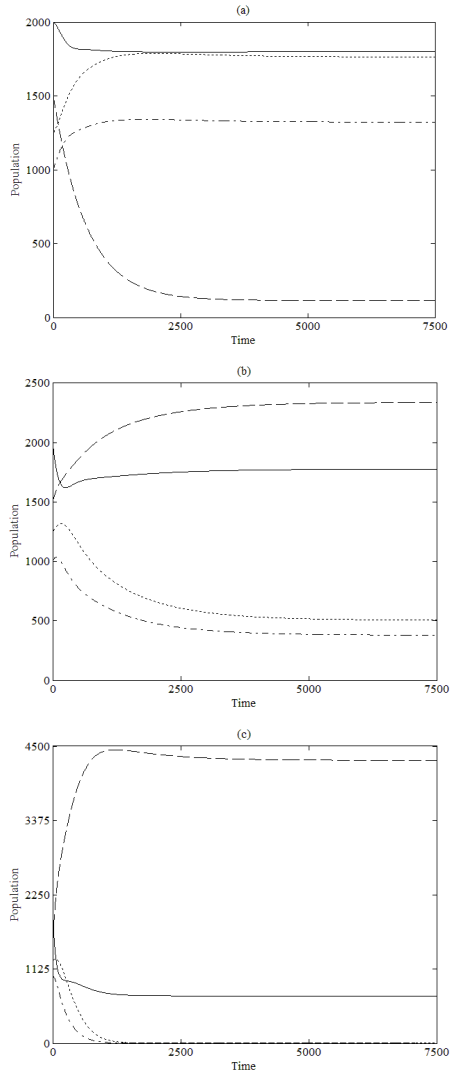
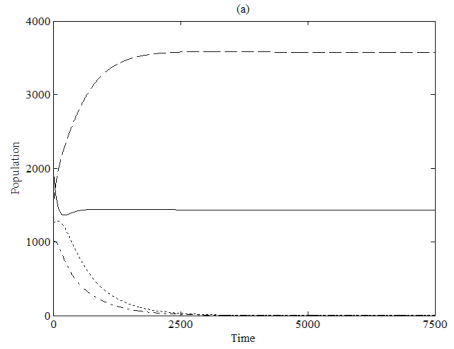


Fig. 5, Time series of the solution of system (2.3). (a) for $\psi = 0.01$, (b) for $\psi = 0.2$, (c) for $\psi = 0.9$.



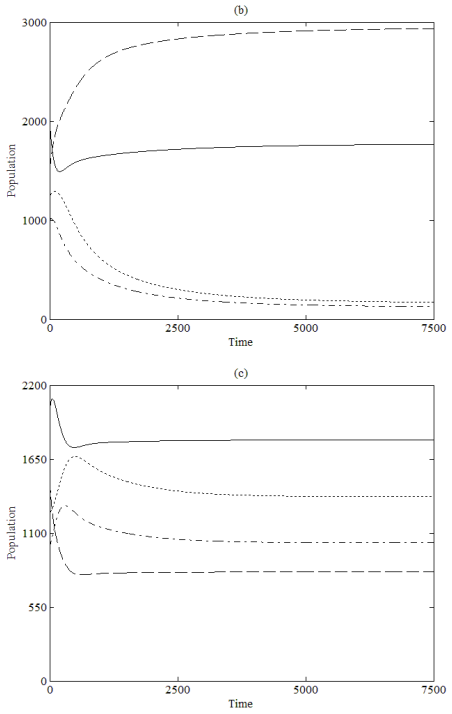
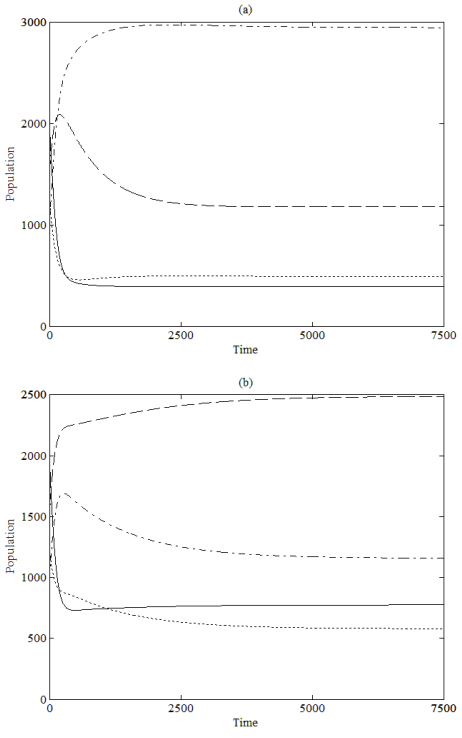


Fig. 6, Time series of the solution of system (2.3). (a) for $\theta = 0.1$, (b) for $\theta = 0.2$, (c) for $\theta = 1$.



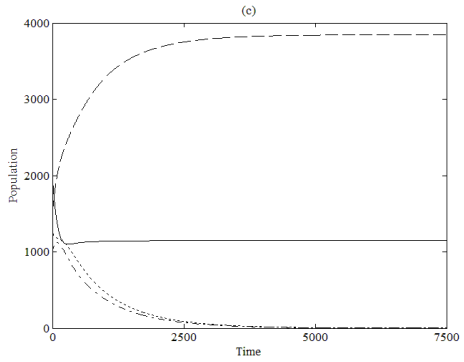


Fig. 7, Time series of the solution of system (2.3). (a) for $\alpha = 0.1$, (b) for $\alpha = 0.3$, (c) for $\alpha = 0.6$.

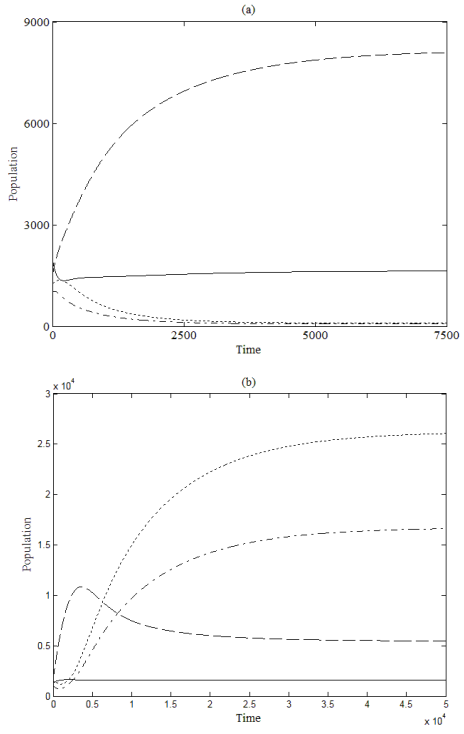


Fig. 8, Time series of the solution of system (2.3) for the data given by (7.11) with varying μ. (a) for $\mu = 0.05$, (b) for $\mu = 0.01$.

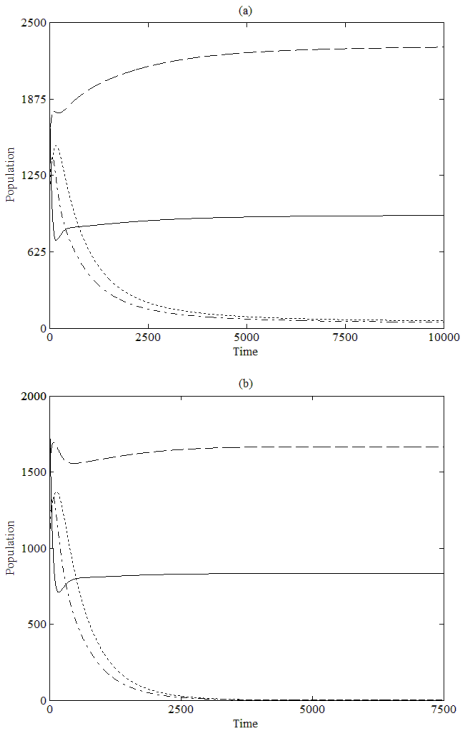


Fig. 9, Time series of the solution of system (2.3) for the data given by (7.11) with $\beta = 0.001$ and varying μ. (a) for $\mu = 0.15$, (b) for $\mu = 0.2$.

In the following the global dynamics of system (7.1) is carried out. System (7.1) is solved numerically for the following set of parameters, which satisfies condition (7.4a), and then the trajectories are drawn in Fig. 10.

$$\Lambda = 400,\ A = 100,\ p = 0,\ \beta = 0.00015,\quad \beta_0 = 0,\ \mu = 0.1, \alpha = 0.8,\ \gamma = 0.5 \qquad (7.12)$$

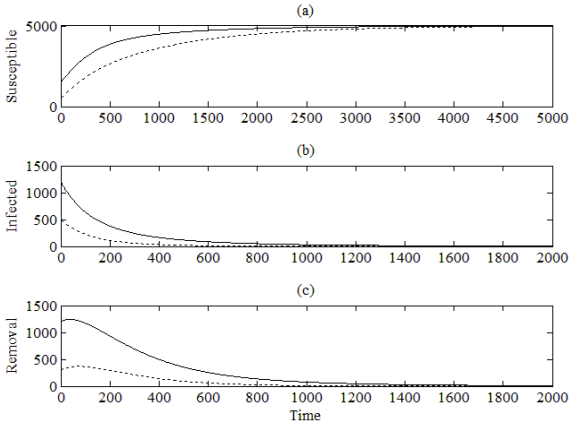


Fig. 10, Time series of the solution of system (7.1). (a) trajectories of S, (b) trajectories of I and (c) trajectories of R, the solid starting at (1500,1200,1200) and dotted starting at (500,500, 300).

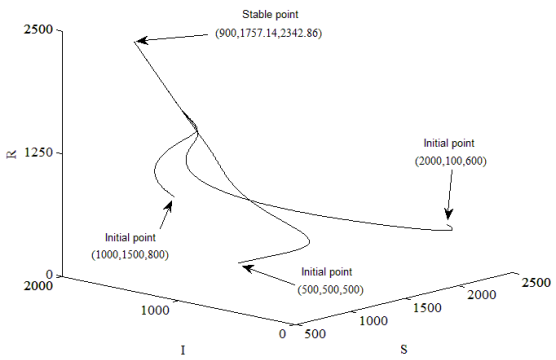


Fig. 11, Phase plot of system (7.1) starting from three different initial points.

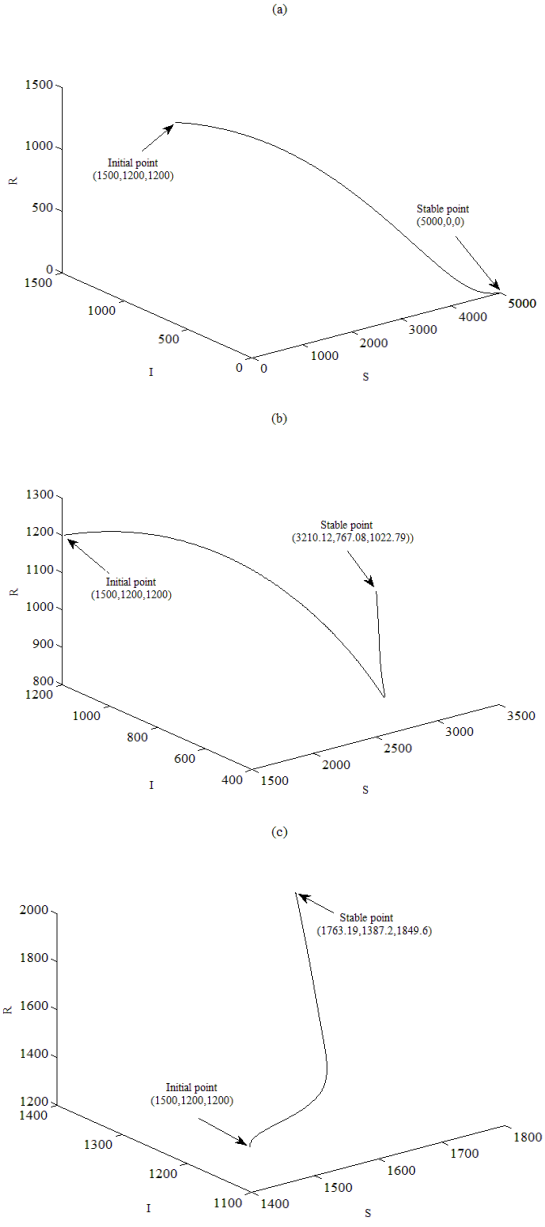


Fig. 12, Phase plot of system (7.1). (a) The solution approaches to (5000, 0, 0) for $\beta_{\circ} = 0$, (b) The solution approaches to (3210.12,767.02,1022.79) for $\beta_{\circ} = 0.1$, (c) The solution approaches to (1763.19,1387.2,1849.6) for $\beta_{\circ} = 0.5$.

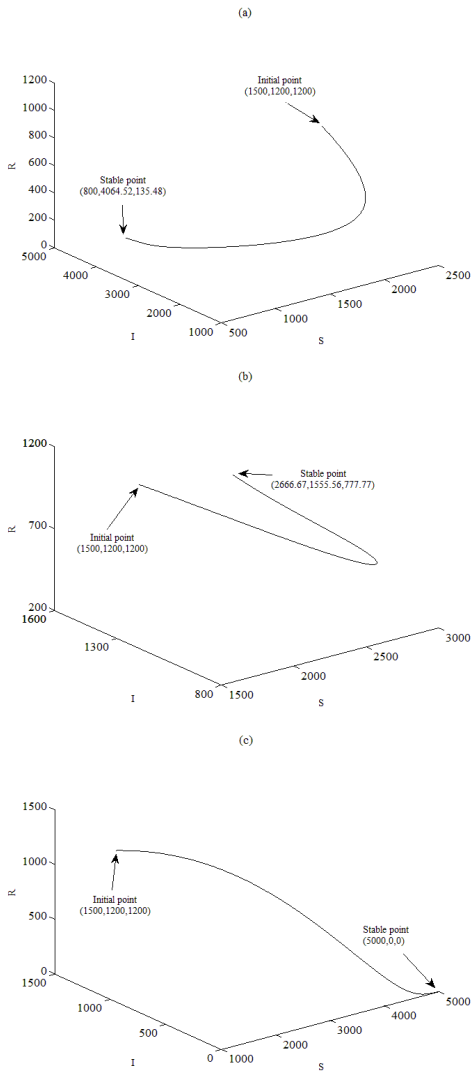


Fig. 13 Phase plot of system (7.1). (a) The solution approaches to (800,4064.52,135.48) for $\alpha = 0.02$, (b) The solution approaches to (2666.67,1555.56,777.77) for $\alpha = 0.3$, (c) The solution approaches to (5000,0,0) for $\alpha = 0.9$.

In **Fig. 2** shows that the solution of system (2.3) approaches asymptotically to the disease free equilibrium point has a globally asymptotically stable disease free $E_{\circ} = (115385,384615,0,0)$ starting from two different initial points and this is confirming our obtained analytical results. **Fig. 3** shows clearly the convergence of system (2.3) to the endemic equilibrium point $E_1 = (871.996,280043,568.96,758.61)$ asymptotically from two different initial points. This is indicates to occurrence of a transcritical bifurcation near the disease free equilibrium point at a specific value of $\beta \in (0.0005 0.001)$, so E_0 became unstable and the solution of system (2.3)

approaches to E_1. In addition to that, the above two figures refer to that increasing the contact rate between S and I causes destabilizing to disease free equilibrium point and the system approaches instead to the endemic point. Note that, in the above figures (4-9), we will use the following representations: Solid line for describing trajectory of **S**; dashed line for describing trajectory of **V**; dash dot line for describing trajectory of **I**; dotted line for describing trajectory of **R** and starting at (2000, 1500, 1000, 1250). In **Fig. 4** as the incidence rate of disease resulting from external sources increases (through increasing β_0), the disease free equilibrium point of system (2.3) becomes unstable point and the trajectory of system (2.3) approaches asymptotically to the endemic equilibrium point. In fact as β_0 increases it is observed that the number of susceptible and vaccinated individuals decrease and the number of recover and infected individuals increases. **Fig. 5** it is clear that as the rate of vaccine coverage increases the endemic equilibrium point of system (2.3) becomes unstable point and the trajectory of the system approaches asymptotically to the disease free equilibrium point attendant that increasing in vaccined individuals and decreasing in susceptible individuals. **Fig. 6** the increasing θ (that is decreasing the lifetime of vaccine immunity) destabilizes the disease free equilibrium point and then the solution of system (2.3) approaches to endemic equilibrium point attended that increasing in the susceptible, infected and recover individuals while the number of vaccinated individuals decreases. From **Fig. 7** that, as the recovery rate increases from 0.1 to 0.6 the endemic equilibrium point of system (2.3) becomes unstable point and the trajectory of system (2.3) approaches asymptotically to the disease free equilibrium point. But the number of susceptible and vaccinated individuals increases while the number of the infected and recover individuals decreases. In **Fig. 8** however, , increases the parameter μ more than 0.1 keeping other parameters fixed as in (7.11) with $\beta = 0.001$ causes transferring in the stability of system (2.3) from endemic equilibrium point to disease free equilibrium point as shown in **Fig. 9**. Therefore, the death rate due to the disease plays a vital role as bifurcation parameter of system (2.3). **Fig. 10** Shows that the solution of system (7.1) approaches asymptotically to the disease free equilibrium point $E_2 = (5000,0,0)$ from two different initial data. **Fig. 11** shows the existence of a unique endemic equilibrium point of system (7.1), which is globally asymptotically stable. **Fig. 12** the external incidence rate increases the disease free equilibrium point of system (7.1) becomes unstable point and the trajectory of system (7.1) approaches asymptotically to the endemic equilibrium point, and then the number of susceptible individuals decrease while the number of the infected and recover individuals increases. **Fig. 13** the recovery rate increases the endemic equilibrium point of system (7.1) becomes unstable point and the trajectory of system (7.1) approaches asymptotically to the disease free equilibrium point attendant that increasing the number of susceptible individuals and decreasing in the numbers of the infected and recover individuals.

9. Conclusion and discussion:

In this paper, two mathematical models have been proposed and analyzed. The objective is to study the effect of immigrants, existence and nonexistence vaccine, and then existence of external sources of the disease in the environment on the dynamical behavior of SVIRS and SIRS epidemic models. The existence and the stability analysis of all possible equilibrium points are studied analytically as well as numerically. It is observed that system (2.3) and system (7.1) have transcritical bifurcation near the disease free equilibrium point, but neither saddle node nor pitchfork bifurcation can accrue. Further both the systems (2.3) and (7.1) do not have Hopf bifurcation near the endemic equilibrium point. Finally according to the numerically simulation the following results are obtained:

1. Both the systems (2.3) and (7.1) do not have periodic dynamic, instead it they approach either to the disease free equilibrium point or else to endemic equilibrium point.
2. As the number of the infected immigrant individuals and the incidence rate of disease (external incidence rate or contact incidence rate) increase, the asymptotic behavior of the systems (2.3) and (7.1) transfer from approaching to disease free equilibrium point to the endemic equilibrium point.
3. As the lifetime of vaccine immunity decreases (the losing vaccine immunity rate (θ) increases), then the disease free equilibrium point of system (2.3) becomes unstable and the solution will approaches to the endemic equilibrium point. Further, similar result is obtained in systems (2.3) and (7.1) when the natural death rate decreases.
4. As the recovery rates in the systems (2.3) and (7.1) increase then the solution in both the systems will be transfer from stability at endemic equilibrium point to stability at disease free equilibrium point. Further, similar result is obtained in case of system (2.3) when the vaccine coverage rate increases.
5. Finally, changing the lifetime of removal individual's immunity in both the system (2.3) and (7.1) do not have vital effect on the dynamical behavior of each of them.

References:

[1] W.O. Kermack, A.G. Mckendrick, A contribution to the mathematical theory of epidemics, Proc. R. Soc. London a 115 (1927) 700-721.

[2] R.M. Anderson, R. M. May, Population Biology of Infectious Disease, Springer – Verlag, New York, (1982).

[3] F. Brauer, C. Castillo-Chavez, Mathematical Models in Population Biology and Epidemiology, Springer, New York, (2001).

[4] O. Diekmann, J. A. P. Heesterbeek, Mathematical Epidemiology of Infectious Disease: Model Building, Analysis and Interpretation, Wiley, New York, (2000).

[5] H.W. Hethcote, The mathematics of infectious disease, SIAM Rev. 42 (2000) 599-653.

[6] Kribs-Zaleta, C.M., Velasco-Hern´andez, J.X.,(2000). A simple vaccination model with multiple endemic states. Math. Biosci. 164: 183-201.

[7] Alexander, M.E., Bowman, C., Moghadas, S.M., Summers, R., Gumel, A.B., Sahai, B.M., A vaccination model for transmission dynamics of influenza. SIAM J. Appl. Dyn. Syst. (2004), 3:503-524.

[8] Shurowq k. Shafeeq, The effect of treatment, immigrants and vaccinated on the dynamic of SIS epidemic model. M.Sc. thesis. Department of Mathematics, College of Science, University of Baghdad. Baghdad, Iraq (2011).

[9] Krishna pada Das, Shovonlal Roy and J. Chattopadhyay, Biosystem. 95 (2009) 188-199.

[10] Sotomayor, J., Generic bifurcations of dynamical systems, in dynamical systems, M. M. Peixoto, New York, academic press (1973).