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The Dynamics Of Four Species Food Web Model With Stage Structure

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ABSTRACT: In this paper, a food web model involving prey-predator system with stage structure in the second level is proposed and studied. The existence, uniqueness and boundedness of the solution of the system are studied. The existence conditions of all possible equilibrium points are determine. The local stability analyses and the regions of global stability of each equilibrium point are investigated. Finally further investigations for the global dynamics of the proposed system are carried out with the help of numerical simulations.

Keywords: ecology model, stage structure, prey-predator model, stability analysis, top predator.

1. Introduction

It is well known that in nature most of the species have two stages, the first is immature and the second stage of mature. Furthermore each species has the ability to interact with any other species in the environment; these interactions differ according to the stage of every species. It is worth to note that, over the last several decades there has been a considerable interest in the study of population dynamics with stage structure. In addition, in each environment there is always a predator residing at the top of a food chain, this type of predator is called top predator or super predator that is meaning that top predator species occupy the highest food level or levels and plays a crucial role in maintaining the health of their ecosystems, see for example [1-4] and the references therein. Most of these studies were focused on prey-predator interactions involving a stage structured predator. Kunal and Milon [5] studied the effect of harvest and bifurcation of a prey–predator model with stage structure. Xiangyun, Jingan and Xueyong [6], proposed and analyzed eco-epidemic model with a stage structure. Tarpon and Charugopal [7] studied a prey-predator model with stage structure for prey. Chen and You [8] studied the permanence, extinction and periodic solution of the periodic predator–prey system with Beddington–DeAngelis functional response and stage structure for prey. They obtained a set of sufficient and necessary conditions which guarantee the permanent of the system. In the last decade, simple multi-species systems comprising of three trophic level food chain were discussed by number of researchers [9–15]. On the other hand, chaos in three species food chain system with classical Lotka–Volterra type interactions and Holling type II functional response was also demonstrated. Hsu et al. [16] studied the three trophic level food chain with ratio-dependent Mechaelis–Menten type of functional response and its applications to biological control. Gakkhar and Naji [17] investigated a three species ratio-dependent food chain. In this paper, however, we will propose and analyze a food web ecosystem involving two competing prey species at the first level; stage structure intermediate predator at the second level, which consumes the prey species at the first level according to Holling type-II functional response; top predator at the third level that preys upon the two competing preys at the first level and the immature predator at the second level while its competes the mature predator at the second level. The local as well as global stability analysis of the modified model is investigated analytically as well as numerically.

2. Model formulation

In this section an ecological system consisting of four species involving stage structure is mathematically formulated and analyzed. It is assumed that these four species are distributed in three levels so that in the first level there are two competing prey species, which are denoted to their population's sizes at time t by $N_1(t)$ and $N_2(t)$ respectively. In the second level there is a stage structure predator species in which $N_3(t)$ denotes to the population size of immature individuals at time t, while $N_4(t)$ denotes to the population size of mature individuals at time t. However, in the third level there is a top predator that denotes to their population size at time t by $N_5(t)$. Finally in order to formulate this system mathematically, the following assumptions are adopted.

1. In the absence of predation, the two competing preys at the first level grow logistically with an intrinsic growth rates $r>0, s>0$ and carrying capacities $K>0, L>0$ for $N_1(t)$ and $N_2(t)$ respectively. However, they are competing each other with intensity of competition rates $a_1>0$ and $a_2>0$ respectively.
2. In case of existence of the predator in the second level it is assumed that the predator is divided into two compartments namely immature predator $N_3(t)$ and mature predator $N_4(t)$. The mature predator consumes the first and second preys according to Lotka-Volterra type of functional response with maximum attack rates $b_1>0$ and $b_2>0$, then the food is up taken by the predator with uptake rates $0<e_1<1$ and $0<e_2<1$ respectively. Moreover, the immature predator can't attack any of the preys, rather than that it depends completely on his parents, so that it feeds on the portion of up taken food by mature predator from the first and second preys with portion rates $0<n<1$ and $0<m<1$ respectively. Finally it is assumed that the immature predator is grown up to be mature with grown up rate $\alpha>0$.

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3. When the top predator exists in the third level, it is assumed that the top predator consumes both the preys in the first level according to Lotka-Volterra type of the functional response with maximum attack rates $c_1 > 0$ and $c_2 > 0$ for $N_1(t)$ and $N_2(t)$ respectively, while it attacks the immature predator at the second level with maximum attack rate $\beta > 0$. Further, it is assumed that there is enter-specific competition between the mature predator and top predator with intensity of competition rates $\gamma_1 > 0$ and $\gamma_2 > 0$ respectively. Finally both the predators (mature predator and top predator) are decay exponentially with natural death rates $d_1 > 0$ and $d_2 > 0$ respectively in the absence of their food.

According to these assumptions the dynamics of the above described food web system can be formulated mathematically with the following set of differential equations:

$$
\begin{aligned}
\frac{dN_1}{dT} &= rN_1\left(1-\frac{N_1}{K}\right) - a_1N_1N_2 - b_1N_1N_4 - c_1N_1N_5 \\
\frac{dN_2}{dT} &= sN_2\left(1-\frac{N_2}{L}\right) - a_2N_1N_2 - b_2N_2N_4 - c_2N_2N_5 \\
\frac{dN_3}{dT} &= ne_1b_1N_1N_4 + me_2b_2N_2N_4 - \alpha N_3 - \beta N_3N_5 \\
\frac{dN_4}{dT} &= \alpha N_3 + (1-n)e_1b_1N_1N_4 + (1-m)e_2b_2N_2N_4 \\
&\quad - \gamma_1N_4N_5 - d_1N_4 \\
\frac{dN_5}{dT} &= e_3c_1N_1N_5 + e_4c_2N_2N_5 + e_5\beta N_3N_5 - \gamma_2N_4N_5 \\
&\quad - d_2N_5
\end{aligned}
\qquad \text{.........(1)}
$$

Here $N_1(0) \ge 0$, $N_2(0) \ge 0$, $N_3(0) \ge 0$, $N_4(0) \ge 0$ and $N_5(0) \ge 0$. Note that the above model contains 23 positive parameters in all, which makes the analysis of the system very difficult. So, in order to reduce the number of parameters and determine which parameters represent the control parameters, the following dimensionless variables are used.

$$
\begin{aligned}
& t = rT,\ x_1 = \frac{N_1}{K},\ x_2 = \frac{a_1}{r}N_2,\ x_3 = \frac{\beta}{r}N_3, \\
& x_4 = \frac{b_1}{r}N_4,\ x_5 = \frac{c_1}{r}N_5,\ u_1 = \frac{s}{r}, u_2 = \frac{r}{a_1L}, \\
& u_3 = \frac{Ka_2}{r},\ u_4 = \frac{b_2}{b_1},\ u_5 = \frac{c_2}{c_1},\ u_6 = \frac{\beta e_1K}{r}, \\
& u_7 = \frac{\beta b_2e_2}{a_1b_1},\ u_8 = \frac{\alpha}{r}, u_9 = \frac{\beta}{c_1},\ u_{10} = \frac{\alpha b_1}{r\beta}, \\
& u_{11} = \frac{Kb_1e_1}{r},\ u_{12} = \frac{b_2e_2}{a_1},\ u_{13} = \frac{\gamma_1}{c_1}, u_{14} = \frac{d_1}{r}, \\
& u_{15} = \frac{c_1Ke_3}{r},\ u_{16} = \frac{c_2e_4}{a_1},\ u_{17} = \frac{\gamma_2}{b_1},\ u_{18} = \frac{d_2}{r}
\end{aligned}
$$

Accordingly, the dimensionless of system (1) becomes

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_1(1-x_1) - x_1x_2 - x_1x_4 - x_1x_5 = f_1(X) \\
\frac{dx_2}{dt} &= u_1x_2(1-u_2x_2) - u_3x_1x_2 - u_4x_2x_4 - u_5x_2x_5 \\
&= f_2(X) \\
\frac{dx_3}{dt} &= nu_6x_1x_4 + mu_7x_2x_4 - u_8x_3 - u_9x_3x_5 = f_3(X) \\
\frac{dx_4}{dt} &= u_{10}x_3 + (1-n)u_{11}x_1x_4 + (1-m)u_{12}x_2x_4 \\
&\quad - u_{13}x_4x_5 - u_{14}x_4 = f_4(X) \\
\frac{dx_5}{dt} &= u_{15}x_1x_5 + u_{16}x_2x_5 + e_5x_3x_5 - u_{17}x_4x_5 - u_{18}x_5 \\
&= f_5(X)
\end{aligned}
\qquad \text{......(2)}
$$

Here $X = (x_1, x_2, x_3, x_4, x_5)^T$, $x_1(0) \ge 0$, $x_2(0) \ge 0$, $x_3(0) \ge 0$, $x_4(0) \ge 0$ and $x_5(0) \ge 0$. Clearly, the interaction functions f_1, f_2, f_3, f_4 and f_5 of system (2) are continuous and have continuous partial derivatives on the state space

$$
R_+^5 = \{X \in R^5 : x_1(0) \ge 0, x_2(0) \ge 0, \\ x_3(0) \ge 0, x_4(0) \ge 0, x_5(0) \ge 0\}.
$$

Hence these functions are Lipschizian on R_+^5 and then the solution of the system (2) with nonnegative initial condition exists and is a unique. Further, all the solutions of system (2) which initiate in R_+^5 are uniformly bounded as shown in the following theorem.

Theorem (1): All the solutions of system (2), which initiate in R_+^5, are uniformly bounded.

Proof: From the first equation of system (2) we get:

$$\frac{dx_1}{dt} \le x_1(1-x_1)$$

Then according to the comparison theorem [18], the above differential inequality gives that

$\limsup_{t\to\infty} x_1(t) \le 1$, hence $x_1(t) \le 1$; $\forall t > 0$

Similarly, from the second equation of system (2) we obtain that

$\limsup_{t\to\infty} x_2(t) \le \frac{1}{u_2}$,hence $x_2(t) \le \frac{1}{u_2}$; $\forall t > 0$

Now define the function $M(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t) + x_5(t)$ and then take the time derivative of $M(t)$ along the solution of system (2) gives:

$$\frac{dM}{dt} \le H - \delta M \implies \frac{dM}{dt} + \delta M \le H$$

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Where $\delta = \min\{1, u_1, \gamma_1 u_{14}, u_{18}\}$, $H = 2x_1 + 2u_1 x_2$ with $\gamma = u_8 - u_{10}$. Now, it is easy to verify that the solution of the above linear differential inequalities can be written

$$M(t) \le \frac{H}{\delta} + \left(M_0 - \frac{H}{\delta}\right) e^{-\delta t}$$

Where $M_0 = (x_1(0), x_2(0), x_3(0), x_4(0), x_5(0))$,So that

$$\limsup_{t\to\infty} M(t) \le \frac{H}{\delta} \implies M(t) \le \frac{H}{\delta};\ \forall t > 0 .$$

Thus all solutions are uniformly bounded and the proof is complete.

■

3. Existence of equilibrium points

It is observed that, system (2) has at most eleven biologically feasible equilibrium points, namely $E_i; i = 0,1,2,\ldots,10$. The existence conditions for each of these equilibrium points are derived in the following. The vanishing equilibrium point $E_0 = (0,0,0,0,0)$ and the axial equilibrium points $E_1 = (1,0,0,0,0)$ and $E_2 = (0, \frac{1}{u_2}, 0,0,0)$ always exist. The first two species equilibrium point $E_3 = (\bar{x}_1, \bar{x}_2, 0,0,0)$, where

$$\bar{x}_1 = \frac{u_1(1-u_2)}{u_3 - u_1 u_2},\ \bar{x}_2 = 1 - \bar{x}_1 \qquad (3a)$$

exists under one set of the following sets of conditions

$$u_3 > u_1 \ \&\ u_2 < 1 \qquad (3b)$$

Or

$$u_3 < u_1 \ \&\ u_2 > 1 \qquad (3c)$$

The second two species equilibrium point $E_4 = (\hat{x}_1, 0,0,0, \hat{x}_5)$, with

$$\hat{x}_1 = \frac{u_{18}}{u_{15}},\ \text{and}\ \hat{x}_5 = 1 - \hat{x}_1 \qquad (4a)$$

exists under the condition

$$u_{18} < u_{15} \qquad (4b)$$

The third two species equilibrium point $E_5 = (0, \breve{x}_2, 0,0, \breve{x}_5)$, where

$$\breve{x}_5 = \frac{u_1}{u_5}(1 - u_2 \breve{x}_2)\ \text{and}\ \breve{x}_2 = \frac{u_{18}}{u_{16}} \qquad (5a)$$

exists under the condition

$$u_2 u_{18} < u_{16} \qquad (5b)$$

Moreover, the first three species equilibrium point $E_6 = (\bar{\bar{x}}_1, 0, \bar{\bar{x}}_3, \bar{\bar{x}}_4, 0)$ where

$$\left.\begin{array}{l} \bar{\bar{x}}_1 = \dfrac{u_8 u_{14}}{n u_6 u_{10} + (1-n) u_8 u_{11}},\ \bar{\bar{x}}_3 = \dfrac{n u_6}{u_8} \bar{\bar{x}}_1 (1 - \bar{\bar{x}}_1) \\ \bar{\bar{x}}_4 = 1 - \bar{\bar{x}}_1 \end{array}\right\} \qquad (6a)$$

exists if the following condition holds

$$u_8 u_{14} < n u_6 u_{10} + (1-n) u_8 u_{11} \qquad (6b)$$

The second three species equilibrium point $E_7 = (0, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, 0)$ where

$$\left.\begin{array}{l} \tilde{x}_2 = \dfrac{u_8 u_{14}}{m u_7 u_{10} + (1-m) u_8 u_{12}}, \\ \tilde{x}_3 = \dfrac{m u_7 u_1}{u_8 u_2} \tilde{x}_2 (1 - u_2 \tilde{x}_2), \\ \tilde{x}_4 = \dfrac{u_1}{u_2}(1 - u_2 \tilde{x}_2) \end{array}\right\} \qquad (7a)$$

exists under the condition

$$u_2 u_8 u_{14} < m u_7 u_{10} + (1-m) u_8 u_{12} \qquad (7b)$$

The third three species equilibrium point $E_8 = (\tilde{\tilde{x}}_1, \tilde{\tilde{x}}_2, 0,0, \tilde{\tilde{x}}_5)$, where

$$\left.\begin{array}{l} \tilde{\tilde{x}}_1 = \dfrac{u_{18} - u_{16} \tilde{\tilde{x}}_2}{u_{15}}, \\ \tilde{\tilde{x}}_2 = \dfrac{(u_3 - u_5) u_{18} + (u_5 - u_1) u_{15}}{(u_3 - u_5) u_{16} + (u_5 - u_1 u_2) u_{15}}, \\ \tilde{\tilde{x}}_5 = 1 - \tilde{\tilde{x}}_1 - \tilde{\tilde{x}}_2 \end{array}\right\} \qquad (8a)$$

exists if the following condition holds

$$0 < \frac{(u_3 - u_5) u_{18} + (u_5 - u_1) u_{15}}{(u_3 - u_5) u_{16} + (u_5 - u_1 u_2) u_{15}} < \frac{u_{15} - u_{18}}{u_{15} - u_{16}} \qquad (8b)$$

The top predator free equilibrium point $E_9 = (\breve{\breve{x}}_1, \breve{\breve{x}}_2, \breve{\breve{x}}_3, \breve{\breve{x}}_4, 0)$,which is given by

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$$\left.\begin{aligned}\breve{\breve{x}}_1 &= \frac{u_8u_{14}(u_4-u_1u_2)-(u_4-u_1)Q_2}{(u_4-u_1u_2)Q_1+(u_3-u_4)Q_2},\\ \breve{\breve{x}}_2 &= \frac{u_4-u_1+(u_3-u_4)\breve{\breve{x}}_1}{(u_4-u_1u_2)},\\ \breve{\breve{x}}_3 &= \frac{1}{u_8}[nu_6\breve{\breve{x}}_1\breve{\breve{x}}_4+mu_7\breve{\breve{x}}_2\breve{\breve{x}}_4],\\ \breve{\breve{x}}_4 &= \frac{u_1-u_1u_2-(u_3-u_1u_2)\breve{\breve{x}}_1}{u_4-u_1u_2}\end{aligned}\right\} \quad (9a)$$

Here $Q_1 = nu_6u_{10}+(1-n)u_8u_{11}$, $Q_2 = mu_7u_{10}+(1-m)u_8u_{12}$. This point exists uniquely in the $Int.R^4_+$ of (x_1,x_2,x_3,x_4) – space provided that one set of the following conditions hold

$$\left.\begin{matrix} u_1u_2<u_4<u_3 & & u_3<u_4<u_1u_2\\ \frac{u_1-u_4}{u_3-u_4}<\breve{\breve{x}}_1<\frac{u_1-u_1u_2}{u_3-u_1u_2} & or & \frac{u_1-u_1u_2}{u_3-u_1u_2}<\breve{\breve{x}}_1<\frac{u_1-u_4}{u_3-u_4}\\ \gamma>0 & & \gamma<0\end{matrix}\right\} \ldots\ldots(9b)$$

with $\gamma = u_8u_{14}(u_4-u_1u_2)-(u_4-u_1)Q_2$.

Finally the positive equilibrium point $E_{10}=(x_1^*,x_2^*,x_3^*,x_4^*,x_5^*)$ of system (2) can be written as:-

$$\left.\begin{aligned} x_1^* &= \frac{1}{R_2}[R_1-R_3x_4^*-R_4x_5^*],\\ ]x_2^* &= \frac{1}{R_2}[L_1+L_2x_4^*+L_3x_5^*],\\ x_3^* &= \frac{x_4^*}{u_{10}R_2}[L_4x_5^*+L_5x_4^*+L_6\end{aligned}\right\} \quad (10a)$$

here $L_1 = u_1-u_3$, $L_2 = u_3-u_4$, $L_3 = u_3-u_5$

$R_1 = u_1u_2-u_1$, $R_2 = u_3-u_5$, $R_3 = u_1u_2-u_4$ and $R_4 = u_1u_2-u_5$. While (x_4^*,x_5^*) represents the unique intersection point in the interior of the positive quadrant of x_4x_5 - plane for the following two isoclines

$$\begin{aligned} f(x_4,x_5) &= r_1x_4^2+r_2x_4+r_3x_5x_4+r_4x_5+r_5\\ g(x_4,x_5) &= s_1x_4x_5^2+(s_2-s_3x_4)x_4x_5+s_4x_4^2+s_5x_4\end{aligned} \quad \ldots\ldots(10b)$$

Where

$$r_1=\frac{L_5e_5}{u_{10}u_{17}R_2}, r_2=\frac{u_{16}L_2}{u_{17}R_2}+\frac{L_6e_5}{u_{10}u_{17}R_2}-\frac{u_{15}R_3}{u_{17}R_2}-1$$

$$r_3=\frac{e_5L_4}{u_{10}u_{17}R_2}, r_4=\frac{u_{16}L_3-u_{15}R_4}{u_{17}R_2},$$

$$r_5=\frac{u_{15}R_1}{u_{17}R_2}+\frac{u_{16}L_1}{u_{17}R_2}-\frac{u_{18}}{u_{17}}, \delta_1=\frac{-u_9L_4}{u_{10}R_2},$$

$$\delta_2 = mu_7\frac{L_3}{R_2}-nu_6\frac{R_4}{R_2}-\frac{u_8L_4}{u_{10}R_2}-\frac{u_9L_6}{u_{10}R_2},$$

$$\delta_3=\frac{u_9L_5}{u_{10}R_2}, \delta_4=-\left[nu_6\frac{R_3}{R_2}+\frac{u_8L_5}{u_{10}R_2}-mu_7\frac{L_2}{R_2}\right]$$

$$\delta_5 = nu_6\frac{R_1}{R_2}+mu_7\frac{L_1}{R_2}-\frac{u_8L_6}{u_{10}R_2}$$

Straightforward computation shows that E_{10} exists uniquely in the $Int.R^5_+$ provided that the following set of conditions hold For the positivety of x_1^*, x_2^* and x_3^* we should have:

$$\left.\begin{matrix}\max\{u_4,u_5\}<u_3<u_1;\ u_2>1;\ \alpha>0;\ \beta>0;\ \varphi>0\\ OR\\ u_1<u_3<\min\{u_4,u_5\};\ u_2<1;\ \alpha<0;\ \beta<0;\ \varphi<0\end{matrix}\right\} \ldots\ldots(10c)$$

However for getting unique positive intersection point (x_4^*,x_5^*) we should have:

$$u_{18}R_2>u_{15}R_1+u_{16}L_1;\ u_{16}L_3>u_{15}R_4 \quad (10d)$$

$$\left.\begin{matrix} r_3x_4+r_4<0\\ OR\\ 2r_1x_4+r_2+r_3x_5<0\end{matrix}\right\} \quad (10e)$$

$$\left.\begin{matrix} 2\delta_1x_4x_5+(\delta_1-\delta_3x_4)x_4>0\\ \delta_1x_5^2+(\delta_2-\delta_3x_4)x_5-\delta_3x_4x_5+2\delta_4x_4+\delta_5>0\end{matrix}\right\} \ldots\ldots(10f)$$

Where $\alpha = u_{13}R_2+(1-n)u_{11}R_4-(1-m)u_{12}L_3$,
$\beta = (1-n)u_{11}R_3-(1-m)u_{12}L_2$
$\varphi = u_{14}R_2-(1-n)u_{11}R_1-(1-m)u_{12}L_1$.

4. The stability analysis

In this section, the local stability of the equilibrium points of system (2) is investigated using the linearization method. It is easy to verify that the Jacobian matrix of system (2), at the general point (x_1,x_2,x_3,x_4,x_5), can be written as

$$J=(d_{ij})_{5\times5};\ i,j=1,2,\cdots,5 \quad (11)$$

Where

$d_{11}=-x_1+[1-x_1-x_2-x_4-x_5]$, $d_{12}=-x_1$, $d_{13}=0$,
$d_{14}=d_{15}=-x_1$, $d_{21}=-u_3x_2$,
$d_{22}=-u_1u_2x_2+[u_1(1-u_2x_2)-u_3x_1-u_4x_4-u_5x_5]$, $d_{23}=0$,
$d_{24}=-u_4x_2$, $d_{25}=-u_5x_2$, $d_{31}=nu_6x_4$, $d_{32}=nu_7x_4$,
$d_{33}=-u_8$
$d_{34}=nu_6x_1+mu_7x_2$, $d_{35}=-u_9x_3$,
$d_{41}=(1-n)u_{11}x_4$, $d_{42}=(1-m)u_{12}x_4$,

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$d_{43}=u_{10}$, $d_{45}=-u_{13}x_4$, $d_{54}=-u_{17}x_5$
$d_{44}=(1-n)u_{11}x_1+(1-m)u_{12}x_2-u_{13}x_5-u_{14}$,
$d_{51}=u_{15}x_5$, $d_{52}=u_{16}x_5$, $d_{53}=e_5x_5$,
$d_{55}=u_{15}x_1+u_{16}x_2+e_5x_3-u_{17}x_4-u_{18}$

Therefore, the Jacobian matrix of system (2) at the vanishing equilibrium point E_0 is:

$$J(E_0)=\begin{pmatrix}1&0&0&0&0\\0&u_1&0&0&0\\0&0&-u_8&0&0\\0&0&u_{10}&-u_{14}&0\\0&0&0&0&-u_{18}\end{pmatrix} \quad \text{.......(12)}$$

Thus the eigenvalues of $J(E_0)$ are:

$\lambda_{x_1}=1>0$, $\lambda_{x_2}=u_1>0$, $\lambda_{x_3}=-u_8<0$, $\lambda_{x_4}=-u_{14}<0$, $\lambda_{x_5}=-u_{18}<0$.

Therefore E_0 is unstable saddle point. The Jacobian matrix of system (2) at E_1 is written by

$$J(E_1)=\begin{pmatrix}-1&-1&0&-1&-1\\0&u_1-u_3&0&0&0\\0&0&-u_8&nu_6&0\\0&0&u_{10}&(1-n)u_{11}-u_{14}&0\\0&0&0&0&u_{15}-u_{18}\end{pmatrix} \quad \text{......(13a)}$$

Accordingly the characteristic equation of $J(E_1)$ can be written as

$$(-1-\lambda)[(u_1-u_3)-\lambda][(u_{15}-u_{18})-\lambda][\lambda^2+B_1\lambda+B_2]=0 \quad \text{....(13b)}$$

where $B_1=-[(1-n)u_{11}-u_{14}-u_8]$ and
$B_2=u_8u_{14}-((1-n)u_{11}u_8+nu_6u_{10})$.

So either

$$(-1-\lambda)[(u_1-u_3)-\lambda][(u_{15}-u_{18})-\lambda]=0$$

Which give the eigenvalues of $J(E_1)$ in the x_1,x_2 and x_5 direction respectively as

$$\lambda_{x_1}=-1,\ \lambda_{x_2}=u_1-u_3,\ \lambda_{x_5}=u_{15}-u_{18} \quad \text{(13c)}$$

or

$$\lambda^2+B_1\lambda+B_2=0$$

Which gives the other two eigenvalues of $J(E_1)$ in the x_3 and x_4 direction as

$$\lambda_{x_3},\lambda_{x_4}=\frac{-B_1}{2}\pm\frac{1}{2}\sqrt{B_1^2-4B_2} \quad \text{(13d)}$$

Therefore, all the above eigenvalues have negative real parts if the following conditions hold

$$\left.\begin{array}{l}u_1<u_3\\u_{15}<u_{18}\\(1-n)u_{11}u_8+nu_6u_{10}<u_8u_{14}\end{array}\right\} \quad \text{(13e)}$$

So, E_1 is locally asymptotically stable in R_+^5. However, it is saddle point otherwise.

The Jacobian matrix of system (2) at E_2 is given by

$$J(E_2)=\begin{pmatrix}1-\frac{1}{u_2}&0&0&0&0\\-\frac{u_3}{u_2}&-u_1&0&-\frac{u_4}{u_2}&-\frac{u_5}{u_2}\\0&0&-u_8&m\frac{u_7}{u_2}&0\\0&0&u_{10}&\frac{(1-m)u_{12}}{u_2}-u_{14}&0\\0&0&0&0&\frac{u_{16}}{u_2}-u_{18}\end{pmatrix} \quad \text{........(14a)}$$

Therefore the characteristic equation is

$$\left[\left(1-\frac{1}{u_2}\right)-\lambda\right][-u_1-\lambda]\left[\left(\frac{u_{16}}{u_2}-u_{18}\right)-\lambda\right][\lambda^2+A_1\lambda+A_2]=0 \quad \text{.......(14b)}$$

Here $A_1=-\left[\frac{(1-m)u_{12}}{u_2}-u_{14}-u_8\right]$ and
$A_2=u_8u_{14}-\left(\frac{(1-m)u_8u_{12}+mu_7u_{10}}{u_2}\right)$. So, either

$$\left[\left(1-\frac{1}{u_2}\right)-\lambda\right][-u_1-\lambda]\left[\left(\frac{u_{16}}{u_2}-u_{18}\right)-\lambda\right]=0$$

Which give the eigenvalues of $J(E_2)$ in the x_1,x_2 and x_5 direction respectively as

$$\lambda_{x_1}=1-\frac{1}{u_2},\ \lambda_{x_2}=-u_1,\ \lambda_{x_5}=\frac{u_{16}-u_2u_{18}}{u_2} \quad \text{.......(14c)}$$

or

$$\lambda^2+A_1\lambda+A_2=0$$

Which gives the other two eigenvalues of $J(E_2)$ in the x_3 and x_4 direction as

$$\lambda_{x_3},\lambda_{x_4}=\frac{-A_1}{2}\pm\frac{1}{2}\sqrt{A_1^2-4A_2} \quad \text{(14d)}$$

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Straightforward computation shows that all the eigenvalues of $J(E_2)$ have negative real parts if the following conditions hold:

$$\left.\begin{array}{l} u_2 < 1 \\ u_{16} < u_2 u_{18} \\ \dfrac{(1-m)u_8 u_{12} + m u_7 u_{10}}{u_2} < u_8 u_{14} \end{array}\right\} \quad (14e)$$

Hence E_2 is locally asymptotically stable. However, it is a saddle point otherwise. The Jacobian matrix of system (2) at E_3 can be written as

$$J(E_3) = \begin{pmatrix} -\bar{x}_1 & -\bar{x}_1 & 0 & -\bar{x}_1 & -\bar{x}_1 \\ -u_3\bar{x}_2 & -u_1 u_2 \bar{x}_2 & 0 & -u_4\bar{x}_4 & -u_5\bar{x}_2 \\ 0 & 0 & -u_8 & n u_6 \bar{x}_1 + m u_7 \bar{x}_2 & 0 \\ 0 & 0 & u_{10} & \bar{d}_{44} & 0 \\ 0 & 0 & 0 & 0 & \bar{d}_{55} \end{pmatrix} \text{.......(15a)}$$
$$= (\bar{d}_{ij})$$

Here

$$\bar{d}_{44} = (1-n)u_{11}\bar{x}_1 + (1-m)u_{12}\bar{x}_2 - u_{14},$$
$$\bar{d}_{55} = u_{15}\bar{x}_1 + u_{16}\bar{x}_2 - u_{18}$$

Then the characteristic equation of $J(E_3)$ is given by

$$\left[\bar{\lambda}^2 + \bar{A}_1\bar{\lambda} + \bar{A}_2\right]\left[\bar{\lambda}^2 + \bar{B}_1\bar{\lambda} + \bar{B}_2\right]\left[(u_{15}\bar{x}_1 + u_{16}\bar{x}_2 - u_{18}) - \bar{\lambda}\right] = 0 \text{ .....(15b)}$$

where

$\bar{A}_1 = \bar{x}_1 + u_1 u_2 \bar{x}_2$ and $\bar{A}_2 = (u_1 u_2 - u_3)\bar{x}_1 \bar{x}_2 > 0$ under the second condition of the existence of E_3. While

$$\bar{B}_1 = -[(1-n)u_{11}\bar{x}_1 + (1-m)u_{12}\bar{x}_2 - u_{14} - u_8],$$
$$\bar{B}_2 = u_8 u_{14} - \gamma_1 \bar{x}_1 - \gamma_2 \bar{x}_2$$

with $\gamma_1 = (1-n)u_8 u_{11} + n u_6 u_{10}$, $\gamma_2 = (1-m)u_8 u_{12} + m u_7 u_{10}$.

Therefore the eigenvalues can be written as:

$$\left.\begin{array}{l} \bar{\lambda}_{x_1}, \bar{\lambda}_{x_2} = \dfrac{-\bar{A}_1}{2} \pm \dfrac{1}{2}\sqrt{\bar{A}_1^2 - 4\bar{A}_2} \\ \bar{\lambda}_{x_3}, \bar{\lambda}_{x_4} = \dfrac{-\bar{B}_1}{2} \pm \dfrac{1}{2}\sqrt{\bar{B}_1^2 - 4\bar{B}_2} \\ \bar{\lambda}_{x_5} = u_{15}\bar{x}_1 + u_{16}\bar{x}_2 - u_{18} \end{array}\right\} \quad (15c)$$

Accordingly, it is easy to verify that all these eigenvalues have negative real parts if the following conditions are satisfied

$$\left.\begin{array}{l} u_{15}\bar{x}_1 + u_{16}\bar{x}_2 < u_{18} \\ \gamma_1 \bar{x}_1 + \gamma_2 \bar{x}_2 < u_8 u_{14} \\ (1-n)u_{11}\bar{x}_1 + (1-m)u_{12}\bar{x}_2 < u_8 + u_{14} \end{array}\right\} \quad (15d)$$

Hence, E_3 is locally asymptotically stable. However, it is a saddle point otherwise.

The Jacobian matrix of system (2) at E_4 can be written as

$$J(E_4) = \begin{pmatrix} -\hat{x}_1 & -\hat{x}_1 & 0 & -\hat{x}_1 & -\hat{x}_1 \\ 0 & \hat{d}_{22} & 0 & 0 & 0 \\ 0 & 0 & -u_8 & u n_6 \hat{x}_1 & 0 \\ 0 & 0 & u_{10} & \hat{d}_{44} & 0 \\ u_{15} - u_{18} & u_{16}\hat{x}_5 & e_5 \hat{x}_5 & -u_{17}\hat{x}_5 & 0 \end{pmatrix} \text{....(16a)}$$
$$= (\hat{d}_{ij})$$

Here $\hat{d}_{22} = u_1 - u_3\hat{x}_1 - u_5\hat{x}_5$ $\hat{d}_{44} = (1-n)u_{11}\hat{x}_1 - u_{13}\hat{x}_5 - u_{14}$

The characteristic equation of $J(E_4)$ is given by

$$(\hat{d}_{22} - \hat{\lambda})\left[\hat{\lambda}^2 + \hat{A}_1\hat{\lambda} + \hat{A}_2\right]\left[\hat{\lambda}^2 + \hat{B}_1\hat{\lambda} + \hat{B}_2\right] = 0 \quad (16b)$$

Where $\hat{A}_1 = \hat{x}_1$ and $\hat{A}_2 = (u_{15} - u_{18})\hat{x}_1$, while $\hat{B}_1 = u_8 + u_{14} + u_{13}\hat{x}_5 - (1-n)u_{11}\hat{x}_1$ and $\hat{B}_2 = u_8 u_{14} + u_8 u_{13}\hat{x}_5 - [(1-n)u_8 u_{11} + n u_6 u_{10}]\hat{x}_1$. Therefore the eigenvalues are:

$$\begin{array}{l} \hat{\lambda}_{x_2} = u_1 - u_3\hat{x}_1 - u_5\hat{x}_5 \\ \hat{\lambda}_{x_1}, \hat{\lambda}_{x_5} = \dfrac{-\hat{A}_1}{2} \pm \dfrac{1}{2}\sqrt{\hat{A}_1^2 - 4\hat{A}_2} \\ \hat{\lambda}_{x_3}, \hat{\lambda}_{x_4} = \dfrac{-\hat{B}_1}{2} \pm \dfrac{1}{2}\sqrt{\hat{B}_1^2 - 4\hat{B}_2} \end{array} \quad (16c)$$

Hence, all these eigenvalues have negative real parts if the following conditions are satisfied

$$\left.\begin{array}{l} u_1 < u_3\hat{x}_1 + u_5\hat{x}_5 \\ [(1-n)u_8 u_{11} + n u_6 u_{10}]\hat{x}_1 < u_8 u_{14} + u_8 u_{13}\hat{x}_5 \end{array}\right\} \quad (16d)$$

Thus, E_4 is locally asymptotically sable in the R_+^5, however, it is a saddle point otherwise.

The Jacobian matrix of system (2) at E_5 can be written as

$$J(E_5) = \begin{pmatrix} 1 - \breve{x}_2 - \breve{x}_5 & 0 & 0 & 0 & 0 \\ -u_3\breve{x}_2 & -u_1 u_2 \breve{x}_2 & 0 & -u_4\breve{x}_2 & -u_5\breve{x}_2 \\ 0 & 0 & -u_8 & m u_7 \breve{x}_2 & 0 \\ 0 & 0 & u_{10} & \breve{d}_{44} & 0 \\ u_{15}\breve{x}_5 & u_{16}\breve{x}_5 & e_5\breve{x}_5 & -u_{17}\breve{x}_5 & 0 \end{pmatrix} \text{.....(17a)}$$
$$= (\breve{d}_{ij})$$

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Here $\breve{d}_{44} = (1-m)u_{12}\breve{x}_2 - u_{13}\breve{x}_5 - u_{14}$. So the characteristic equation of $J(E_5)$ is given by:

$$\left[\breve{d}_{11} - \breve{\lambda}\right]\left[\breve{\lambda}^2 + \breve{A}_1\breve{\lambda} + \breve{A}_2\right]\left[\breve{\lambda}^2 + \breve{B}_1\breve{\lambda} + \breve{B}_2\right] = 0 \quad \text{(17b)}$$

Where $\breve{A}_1 = u_1 u_2 \breve{x}_2$ and $\breve{A}_2 = u_5 u_{16} \breve{x}_2 \breve{x}_5$, while $\hat{B}_1 = u_8 + u_{14} + u_{13}\breve{x}_5 - (1-m)u_{12}\breve{x}_2$, $\hat{B}_2 = u_8 u_{14} + u_8 u_{13}\breve{x}_5 - [(1-m)u_8 u_{12} + m u_7 u_{10}]\breve{x}_2$. Thus the eigenvalues of $J(E_5)$ can be written as:

$$\left.\begin{aligned}
&\breve{\lambda}_{x_1} = 1 - \breve{x}_2 - \breve{x}_5 \\
&\breve{\lambda}_{x_2}, \breve{\lambda}_{x_5} = \frac{-\breve{A}_1}{2} \pm \frac{1}{2}\sqrt{\breve{A}_1^2 - 4\breve{A}_2} \\
&\breve{\lambda}_{x_3}, \breve{\lambda}_{x_4} = \frac{-\breve{B}_1}{2} \pm \frac{1}{2}\sqrt{\breve{B}_1^2 - 4\breve{B}_2}
\end{aligned}\right\} \quad \text{(17c)}$$

Now straightforward computation shows that all the eigenvalues of $J(E_5)$ have negative real parts provided that the following conditions are satisfied

$$\left.\begin{aligned}
&1 < \breve{x}_2 + \breve{x}_5 \\
&\left[(1-m)u_8 u_{12} + m u_7 u_{10}\right]\breve{x}_2 < u_8 u_{14} + u_8 u_{13}\breve{x}_5
\end{aligned}\right\} \quad \text{(17d)}$$

Hence E_5 is locally asymptotically stable in the R_+^5, however it is a saddle point otherwise. The Jacobian matrix of system (2) at E_6 can be written as

$$J(E_6) = \begin{pmatrix}
-\bar{\bar{x}}_1 & -\bar{\bar{x}}_1 & 0 & -\bar{\bar{x}}_1 & -\bar{\bar{x}}_1 \\
0 & \bar{\bar{d}}_{22} & 0 & 0 & 0 \\
n u_6 \bar{\bar{x}}_4 & m u_7 \bar{\bar{x}}_4 & -u_8 & n u_6 \bar{\bar{x}}_1 & -u_9 \bar{\bar{x}}_3 \\
(1-n)u_{11}\bar{\bar{x}}_4 & (1-m)u_{12}\bar{\bar{x}}_4 & u_{10} & \bar{\bar{d}}_{44} & -u_{13}\bar{\bar{x}}_4 \\
0 & 0 & 0 & 0 & \bar{\bar{d}}_{55}
\end{pmatrix} \text{....(18a)}$$
$$= (\bar{\bar{d}}_{ij})$$

Here
$\bar{\bar{d}}_{22} = u_1 - u_3\bar{\bar{x}}_1 - u_4\bar{\bar{x}}_4, \bar{\bar{d}}_{44} = (1-n)u_{11}\bar{\bar{x}}_1 - u_{14}$
$\bar{\bar{d}}_{55} = u_{15}\bar{\bar{x}}_1 + e_5\bar{\bar{x}}_3 - u_{17}\bar{\bar{x}}_4 - u_{18}$.

Hence the characteristic equation of $J(E_6)$ is given by

$$(\bar{\bar{d}}_{22} - \bar{\bar{\lambda}})(\bar{\bar{d}}_{55} - \bar{\bar{\lambda}})[\bar{\bar{\lambda}}^3 + \bar{\bar{A}}_1\bar{\bar{\lambda}}^2 + \bar{\bar{A}}_2\bar{\bar{\lambda}} + \bar{\bar{A}}_3] = 0 \text{ ……(18b)}$$

Where

$\bar{\bar{A}}_1 = -(\bar{\bar{d}}_{11} + \bar{\bar{d}}_{33} + \bar{\bar{d}}_{44})$; $\bar{\bar{A}}_2 = \bar{\bar{d}}_{11}\bar{\bar{d}}_{33} + \bar{\bar{R}}_1 + \bar{\bar{R}}_2$;
$\bar{\bar{A}}_3 = -[\bar{\bar{d}}_{11}\bar{\bar{R}}_2 + \bar{\bar{d}}_{14}\bar{\bar{R}}_3]$

With

$\bar{\bar{R}}_1 = \bar{\bar{d}}_{11}\bar{\bar{d}}_{44} - \bar{\bar{d}}_{14}\bar{\bar{d}}_{41}$; $\bar{\bar{R}}_2 = \bar{\bar{d}}_{33}\bar{\bar{d}}_{44} - \bar{\bar{d}}_{34}\bar{\bar{d}}_{43}$;
$\bar{\bar{R}}_3 = \bar{\bar{d}}_{31}\bar{\bar{d}}_{43} - \bar{\bar{d}}_{33}\bar{\bar{d}}_{41}$

While

$$\bar{\bar{\Delta}} = \bar{\bar{A}}_1\bar{\bar{A}}_2 - \bar{\bar{A}}_3 = \bar{\bar{A}}_1(\bar{\bar{d}}_{11}\bar{\bar{d}}_{33} + \bar{\bar{R}}_1) - (\bar{\bar{d}}_{33} + \bar{\bar{d}}_{44})\bar{\bar{R}}_2 + \bar{\bar{d}}_{14}\bar{\bar{R}}_3$$

So the eigenvalues in the x_2 and x_5-directions are given by

$$\left.\begin{aligned}
&\bar{\bar{\lambda}}_{x_2} = u_1 - u_3\bar{\bar{x}}_1 - u_4\bar{\bar{x}}_4; \\
&\bar{\bar{\lambda}}_{x_5} = u_{15}\bar{\bar{x}}_1 + e_5\bar{\bar{x}}_3 - u_{17}\bar{\bar{x}}_4 - u_{18}
\end{aligned}\right\} \quad \text{(18c)}$$

However the other three eigenvalues represent the roots of the third order polynomial in Eq. (18b), which have negative real parts if and only if $\bar{\bar{A}}_1 > 0$, $\bar{\bar{A}}_3 > 0$ and $\bar{\bar{\Delta}} > 0$. So straightforward computation shows that all the eigenvalues of $J(E_6)$ have negative real parts if the following conditions are satisfied:

$$\left.\begin{aligned}
&u_1 < u_3\bar{\bar{x}}_1 + u_4\bar{\bar{x}}_4 \\
&u_{15}\bar{\bar{x}}_1 + e_5\bar{\bar{x}}_3 < u_{17}\bar{\bar{x}}_4 + u_{18} \\
&\bar{\bar{x}}_1 < \min\left\{\frac{u_{14}}{(1-n)u_{11}}, \frac{u_8 u_{14}}{(1-n)u_8 u_{11} + n u_6 u_{10}}\right\} \\
&\bar{\bar{A}}_1(\bar{\bar{d}}_{11}\bar{\bar{d}}_{33} + \bar{\bar{R}}_1) - (\bar{\bar{d}}_{33} + \bar{\bar{d}}_{44})\bar{\bar{R}}_2 > \bar{\bar{d}}_{14}\bar{\bar{R}}_3
\end{aligned}\right\} \text{…..(18d)}$$

So, E_6 is locally asymptotically stable, however, it is saddle point otherwise. The Jacobian matrix of system (2) at E_7 can be written as

$$J(E_7) = \begin{pmatrix}
1 - \tilde{x}_2 - \tilde{x}_4 & 0 & 0 & 0 & 0 \\
-u_3\tilde{x}_2 & -u_1 u_2\tilde{x}_2 & 0 & -u_4\tilde{x}_2 & -u_5\tilde{x}_2 \\
n u_6\tilde{x}_4 & m u_7\tilde{x}_4 & -u_8 & m u_7\tilde{x}_2 & -u_9\tilde{x}_3 \\
(1-n)u_{11}\tilde{x}_4 & (1-m)u_{12}\tilde{x}_4 & u_{10} & & -u_{13}\tilde{x}_4 \\
0 & 0 & 0 & 0 & \tilde{d}_{55}
\end{pmatrix} \text{…..(19a)}$$
$$= (\tilde{d}_{ij})$$

Here
$\tilde{d}_{44} = (1-m)u_{12}\hat{x}_2 - u_{14}$ $\tilde{d}_{55} = u_{16}\hat{x}_2 + e_5\hat{x}_3 - u_{17}\tilde{x}_4 - u_{18}$.

The characteristic equation of $J(E_7)$ is written as:

$$(\tilde{d}_{11} - \tilde{\lambda})(\tilde{d}_{55} - \tilde{\lambda})[\tilde{\lambda}^3 + \tilde{A}_1\tilde{\lambda}^2 + \tilde{A}_2\tilde{\lambda} + \tilde{A}_3] = 0 \text{ ……(19b)}$$

Where
$\tilde{A}_1 = -(\tilde{d}_{22} + \tilde{d}_{33} + \tilde{d}_{44})$; $\tilde{A}_2 = \tilde{d}_{22}\tilde{d}_{33} + \tilde{R}_1 + \tilde{R}_2$;
$\tilde{A}_3 = -[\tilde{d}_{22}\tilde{R}_2 + \tilde{d}_{24}\tilde{R}_3]$
$\tilde{R}_1 = \tilde{d}_{22}\tilde{d}_{44} - \tilde{d}_{24}\tilde{d}_{42}$; $\tilde{R}_2 = \tilde{d}_{33}\tilde{d}_{44} - \tilde{d}_{34}\tilde{d}_{43}$;
$\tilde{R}_3 = \tilde{d}_{32}\tilde{d}_{43} - \tilde{d}_{33}\tilde{d}_{42}$

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While
$\tilde{\Delta} = \tilde{A}_1\tilde{A}_2 - \tilde{A}_3 = \tilde{A}_1[\tilde{d}_{22}\tilde{d}_{33} + \tilde{R}_1] - (\tilde{d}_{33} + \tilde{d}_{44})\tilde{R}_2 + \tilde{d}_{24}\tilde{R}_3$

Therefore the eigenvalues in the x_1 and x_5-directions are given by

$$\left.\begin{aligned} \tilde{\lambda}_{x_1} &= 1 - \tilde{x}_2 - \tilde{x}_4 , \\ \tilde{\lambda}_{x_5} &= u_{16}\tilde{x}_2 + e_5\tilde{x}_3 - u_{17}\tilde{x}_4 - u_{18} \end{aligned}\right\} \qquad (19c)$$

However the other three eigenvalues represent the roots of the third order polynomial in Eq. (19b), which have negative real parts if and only if $\tilde{A}_1 > 0$, $\tilde{A}_3 > 0$ and $\tilde{\Delta} > 0$. So straightforward computation shows that all the eigenvalues of $J(E_7)$ have negative real parts if the following conditions are satisfied:

$$\left.\begin{aligned} & 1 < \tilde{x}_2 + \tilde{x}_4 \\ & u_{16}\tilde{x}_2 + e_5\tilde{x}_3 < u_{17}\tilde{x}_4 + u_{18} \\ & \tilde{x}_2 < \min\left\{\frac{u_{14}}{(1-m)u_{12}}, \frac{u_8 u_{14}}{(1-m)u_8 u_{12} + m u_7 u_{10}}\right\} \\ & \tilde{d}_{24}\tilde{R}_3 < \tilde{A}_1[\tilde{d}_{22}\tilde{d}_{33} + \tilde{R}_1] - (\tilde{d}_{33} + \tilde{d}_{44})\tilde{R}_2 \end{aligned}\right\} \ldots\ldots(19d)$$

So, E_7 is locally asymptotically stable, however, it is saddle point otherwise. Now, since the stability analysis of the remaining equilibrium points of system (2), using linearization method, became more complicated, therefore we will study them with the help of Lyapunov method. In the following we will start first to specify the region of global stability of the equilibrium points $E_i; i = 1,2,\cdots,7$.

Theorem (2): Assume that E_1 is locally asymptotically stable in R_+^5 and the following conditions hold

$$m(1-n)e_5u_5u_7u_{11} + (1-m)u_5u_{12}(u_9u_{15} - nu_6e_5) < (1-n)u_4u_{16}u_9u_{11} \quad \ldots..(20a)$$

$$u_9u_{10}u_{15} < (1-n)e_5u_8u_{11} + ne_5u_6u_{10} \qquad (20b)$$

$$ne_5u_6u_{14} + (1-n)u_9u_{11}u_{15} < u_9u_{14}u_{15} \qquad (20c)$$

$$\frac{\eta_1}{4\eta_2} < (x_1 - 1)^2 \qquad (20d)$$

Where $\eta_1 = \dfrac{u_5u_{15} + u_1u_{16}}{u_5u_{15}}$ and $\eta_2 = \dfrac{u_1u_2u_{16}}{u_5u_{15} + u_1u_{16}}$. Then the equilibrium point E_1 is globally asymptotically stable.

Proof: Consider the following function

$$L_1(x_1, x_2 \ldots, x_5) = r_1(x_1 - 1 - \ln x_1) + r_2x_2 + r_3x_3 + r_4x_4 + r_5x_5$$

Here $r_i; i = 1,2,\cdots,5$ are positive constants to be determined. It is easy to see that $L_1(x_1, x_2 \ldots, x_5) \in C^1(R_+^5, R)$, in addition $L_1(1,0,0,0,0) = 0$, while $L_1(x_1, x_2 \ldots, x_5) > 0$, $\forall (x_1, \ldots, x_5) \in R_+^5$ and $(x_1, \ldots, x_5) \neq (1,0,0,0,0)$. Further more by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned} \frac{dL_1}{dt} = & -r_1(x_1 - 1)^2 - (r_1 + r_2u_3)x_1x_2 + (r_1 + r_2u_1)x_2 \\ & - r_2u_1u_2x_2^2 - [r_1 - r_3nu_6 - r_4(1-n)u_{11}]x_1x_4 \\ & - [r_2u_4 - r_3mu_7 - r_4(1-m)u_{12}]x_2x_4 \\ & - (r_2u_5 - r_5u_{16})x_2x_5 - (r_3u_8 - r_4u_{10})x_3 \\ & - (r_3u_9 - r_5e_5)x_3x_5 - (r_4u_{14} - r_1)x_4 \\ & - (r_4u_{13} + r_5u_{17})x_4x_5 - (r_1 - r_5u_{15})x_1x_5 - r_5u_{18}x_5 \end{aligned}$$

Now by choosing the positive constants $r_i; i = 1,2,\cdots,5$ as follows

$$r_1 = 1,\ r_2 = \frac{u_{16}}{u_5u_{15}},\ r_3 = \frac{e_5}{u_9u_{15}},$$
$$r_4 = \frac{u_9u_{15} - nu_6e_5}{(1-n)u_9u_{11}u_{15}},\ r_5 = \frac{1}{u_{15}}$$

and then substituting them in the above equation , we get

$$\begin{aligned} \frac{dL_1}{dt} = & -(x_1 - 1)^2 - \left(1 + \frac{u_3u_{16}}{u_5u_{15}}\right)x_1x_2 + \eta_1\, x_2\, [1 - \eta_2\, x_2] \\ & - \left[\frac{u_{13}(u_9u_{15} - ne_5u_6)}{(1-n)u_9u_{11}u_{15}} + \frac{u_{17}}{u_{15}}\right]x_4x_5 - u_5u_{18}x_5 \\ & - \left[\frac{(1-n)u_{11}(u_4u_9u_{16} - me_5u_5u_7) - (1-m)u_{12}(u_9u_{15} - nu_6e_5)u_5}{(1-n)u_5u_9u_{11}u_{15}}\right]x_2x_4 \\ & - \left[\frac{e_5u_8}{u_9u_{15}} - \frac{u_{10}(u_9u_{15} - nu_6u_5)}{(1-n)u_9u_{11}u_{15}}\right]x_3 \\ & - \left[\frac{u_{14}(u_9u_{15} - ne_5u_6)}{(1-n)u_9u_{11}u_{15}} - 1\right]x_4 \end{aligned}$$

Now, due to the boundedness of the logistic term $\eta_1x_2[1 - \eta_2x_2]$ by the $\eta_1/4\eta_2$, then its easy to verify that $\dfrac{dL_1}{dt}$ is negative definite under the sufficient conditions (20a)-(20d). Hence the solution of system (2) will approach asymptotically to E_1 from any initial point satisfies the above condition and then the proof is complete.

■

Theorem(3): Assume that $E_2 = (0, \hat{\hat{x}}_2, 0,0,0)$;

$\hat{\hat{x}}_2 = \dfrac{1}{u_2}$ islocally asymptotically stable in R_+^5 then it is a globally asymptotically stable provided that the following

$$m(1-n)e_5u_5u_7u_{11} + (1-m)u_5u_{12}(u_9u_{15} - nu_6e_5) < u_4u_{16}(1-n)u_9u_{11} \quad \ldots\ldots..(21a)$$

$$u_9u_{10}u_{15} < (1-n)e_5u_8u_{11} + ne_5u_6u_{10} \qquad (21b)$$

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$$ne_5u_5u_6u_{14} + (1-n)u_4u_9u_{11}u_{16}\hat{\hat{x}}_2 < u_5u_9u_{14}u_{15} \qquad (21c)$$

$$u_{16}\hat{\hat{x}}_2 < u_{18} \qquad (21d)$$

$$\frac{\beta_1}{4\beta_2} < \frac{u_1u_2u_{16}}{u_5u_{15}} \qquad (21e)$$

Here

$$\beta_1 = \frac{u_5u_{15} + u_3u_{16}\hat{\hat{x}}_2}{u_5u_{15}} \text{ and } \beta_2 = \frac{u_5u_{15}}{u_5u_{15} + u_3u_{16}\hat{\hat{x}}_2}.$$

Proof: Consider the following functions

$$L_2(x_1, x_2, \ldots\ldots, x_5) = c_1x_1 + c_2\left(x_2 - \hat{\hat{x}}_2 - \hat{\hat{x}}_2 \ln\frac{x_2}{\hat{\hat{x}}_2}\right) + c_3x_3 + c_4x_4 + c_5x_5$$

Where $c_i, i = 1,\ldots,5$ are positive constants to be determined. It is easy to see that $L_2(x_1,\ldots,x_5) \in C^1(R_+^5, R)$, and $L_2(0,\hat{\hat{x}}_2,0,0,0) = 0$ while $L_2(x_1,\ldots,x_5) > 0;\ \forall(x_1,\ldots,x_5) \in R_+^5$ and $(x_1,\ldots,x_5) \neq (0,\hat{\hat{x}}_2,0,0,0)$. Further more by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned}\frac{dL_2}{dt} &= c_1x_1(1-x_1) - c_2u_1u_2\left(x_2 - \hat{\hat{x}}_2\right)^2 - c_4u_{14}x_4 \\ &- (c_1 + c_2u_3)x_1x_2 - [c_1 - c_3nu_6 - c_4(1-n)u_{11}]x_1x_4 \\ &- (c_1 - c_5u_{15})x_1x_5 - (c_3u_9 - c_5e_5)x_3x_5 \\ &- (c_3u_8 - c_4u_{10})x_3 + c_2u_3x_1\hat{\hat{x}}_2 \\ &- [c_2u_4 - c_3mu_7 - c_4(1-m)u_{12}]x_2x_4 \\ &- (c_2u_5 - c_5u_{16})x_2x_5 + c_2u_4\hat{\hat{x}}_2x_4 + c_2u_5\hat{\hat{x}}_2x_5 \\ &- (c_4u_{13} + c_5u_{17})x_4x_5 - c_5u_{18}x_5\end{aligned}$$

So by choosing the constants $c_i, i = 1,2,\ldots,5$ as follow

$$c_1 = 1,\quad c_2 = \frac{u_{16}}{u_5u_{15}},\quad c_3 = \frac{e_5}{u_9u_{15}},$$
$$c_4 = \frac{u_9u_{15} - ne_5u_6}{(1-n)u_9u_{11}u_{15}},\quad c_5 = \frac{1}{u_{15}}$$

Thus by substituting these constants in the above equation, we get that

$$\begin{aligned}\frac{dL_2}{dt} &= \beta_1x_1[1-\beta_2x_1] - \frac{u_1u_2u_{16}}{u_5u_{15}}\left(x_2 - \hat{\hat{x}}_2\right)^2 \\ &- \left[\frac{u_5u_{14}(u_9u_{15} - ne_5u_6) - u_4u_9u_{11}u_{16}(1-n)\hat{\hat{x}}_2}{(1-n)u_5u_9u_{11}u_{15}}\right]x_4 \\ &- \left[\frac{(1-n)u_{11}(u_4u_9u_{16} - me_5u_5u_7) - (1-m)u_{12}(u_9u_{15} - ne_5u_6)u_5}{(1-n)u_5u_9u_{11}u_{15}}\right]x_2x_4 \\ &- \left[\frac{(1-n)e_5u_8u_{11} - u_{10}(u_9u_{15} - ne_5u_6)}{(1-n)u_9u_{11}u_{15}}\right]x_3 \\ &- \left[\frac{u_{13}(u_9u_{15} - ne_5u_6)}{(1-n)u_9u_{11}u_{15}} + \frac{u_{17}}{u_{15}}\right]x_4x_5 \\ &- \left(1 + \frac{u_3u_{16}}{u_5u_{15}}\right)x_1x_2 - \left[\frac{u_{18} - u_{16}\hat{\hat{x}}_2}{u_{15}}\right]x_5\end{aligned}$$

Now, due to the boundedness of the logistic term $\beta_1x_1[1-\beta_2x_1]$ by the $\beta_1/4\beta_2$, then its easy to verify that $\frac{dL_2}{dt}$ is negative definite under the sufficient conditions (21a)-(21e). Hence the solution of system (2) will approach asymptotically to E_2 from any initial point satisfies the above condition and then the proof is complete.

■

Theorem (4): Assume that E_3 is locally asymptotically stable in R_+^5. Then, it is a globally asymptotically stable provided that the following conditions hold.

$$m(1-n)e_5u_5u_7u_{11} + (1-m)u_5u_9u_{12}u_{15} < (1-n)u_4u_9u_{11}u_{16} + n(1-m)e_5u_5u_6u_{12} \qquad \ldots\ldots(22a)$$

$$(nu_5u_6) + \frac{(1-n)u_9u_{11}u_{15}\bar{x}_1 + (1-n)u_4u_6u_9u_{11}\bar{x}_2}{u_{14}} < u_9u_{15} < ne_5u_6 + \frac{(1-n)e_5u_8u_{11}}{u_{10}} \qquad (22b)$$

$$u_{15}\bar{x}_1 + u_{16}\bar{x}_2 < u_{18} \qquad (22c)$$

$$\left(u_{15} + \frac{u_3u_{16}}{u_5}\right)^2 < 4\frac{u_1u_2u_{15}u_{16}}{u_5} \qquad (22d)$$

Proof: Consider the following functions

$$L_3(x_1,\ldots,x_5) = \bar{c}_1\left(x_1 - \bar{x}_1 - \bar{x}_1\ln\frac{x_1}{\bar{x}_1}\right) + \bar{c}_2\left(x_2 - \bar{x}_2 - \bar{x}_2\ln\frac{x_1}{\bar{x}_2}\right) + \bar{c}_3x_3 + \bar{c}_4x_4 + \bar{c}_5x_5$$

Where $\bar{c}_i, i = 1,\ldots,5$ are positive constants to be determined. It is easy to verify that $L_3(x_1,\ldots,x_5) \in C^1(R_+^5, R)$, and $L_3(\bar{x}_1,\bar{x}_2,0,0,0) = 0$ while $L_3(x_1,\ldots,x_5) > 0$ for all $(x_1,\ldots,x_5) \in R_+^5$ and $(x_1,\ldots,x_5) \neq (\bar{x}_1,\bar{x}_2,0,0,0)$. Moreover by taking the derivative with respect to the time and simplifying the resulting terms, we get that

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$$\begin{aligned}\frac{dL_3}{dt} &= -\bar{c}_1(x_1-\bar{x}_1)^2 - (\bar{c}_1+\bar{c}_2u_3)(x_1-\bar{x}_1)(x_2-\bar{x}_2)\\ &\quad -\bar{c}_2u_1u_2(x_2-\bar{x}_2)^2 - [\bar{c}_2u_5-\bar{c}_5u_{16}]x_2x_5\\ &\quad -[\bar{c}_1-\bar{c}_3nu_6-\bar{c}_4(1-n)u_{11}]x_1x_4\\ &\quad -[\bar{c}_2u_4-\bar{c}_3mu_7-\bar{c}_4(1-m)u_{12}]x_2x_4\\ &\quad -(\bar{c}_3u_8-\bar{c}_4u_{10})x_3 - [\bar{c}_3u_9-\bar{c}_5e_5]x_3x_5\\ &\quad -[\bar{c}_4u_{13}+\bar{c}_5u_{17}]x_4x_5 - [\bar{c}_1-\bar{c}_5u_{15}]x_1x_5\\ &\quad -[\bar{c}_5u_{18}-\bar{c}_1\bar{x}_1-\bar{c}_2u_5\bar{x}_2]x_5\\ &\quad -[\bar{c}_4u_{14}-\bar{c}_1\bar{x}_1-\bar{c}_2u_4\bar{x}_2]x_4\end{aligned}$$

So by choosing the positive constants as below

$$\bar{c}_1 = u_{15},\quad \bar{c}_2 = \frac{u_{16}}{u_5},\quad \bar{c}_3 = \frac{e_5}{u_9},$$
$$\bar{c}_4 = \frac{u_9u_{15}-ne_5u_6}{(1-n)u_9u_{11}},\quad \bar{c}_5 = 1$$

and then substituting these constants in the above equation, we get that

$$\begin{aligned}\frac{dL_3}{dt} &= -u_{15}(x_1-\bar{x}_1)^2 - \frac{u_1u_2u_{16}}{u_5}(x_2-\bar{x}_2)^2\\ &\quad -\left(u_{15}+\frac{u_3u_{16}}{u_5}\right)(x_1-\bar{x}_1)(x_2-\bar{x}_2)\\ &\quad -\left[\frac{(1-n)u_{11}(u_4u_9u_{16}-me_5u_5u_7)-(1-m)u_5u_{12}(u_9u_{15}-ne_5u_6)}{(1-n)u_5u_9u_{11}}\right]x_2x_4\\ &\quad -\left[\frac{(1-n)e_5u_8u_{11}-u_{10}(u_9u_{15}-ne_5u_6)}{(1-n)u_9u_{11}}\right]x_3\\ &\quad -\left[\frac{u_5u_{14}(u_9u_{15}-ne_5u_6)-(1-n)u_9u_{11}(u_5u_{15}\bar{x}_1+u_4u_{16}\bar{x}_2)}{(1-n)u_5u_9u_{11}}\right]x_4\\ &\quad -\left[\frac{u_{13}(u_9u_{15}-ne_5u_6)}{(1-n)u_9u_{11}}+u_{17}\right]x_4x_5\\ &\quad -[u_{18}-u_{15}\bar{x}_1-u_{16}\bar{x}_2]x_5\end{aligned}$$

So, by using condition (22d) we obtain that

$$\begin{aligned}\frac{dL_3}{dt} &< -\left[\sqrt{u_{15}}(x_1-\bar{x}_1)+\sqrt{\frac{u_1u_2u_{16}}{u_5}}(x_2-\bar{x}_2)\right]^2\\ &\quad -\left[\frac{(1-n)u_{11}(u_4u_9u_{16}-me_5u_5u_7)-(1-m)u_5u_{12}(u_9u_{15}-ne_5u_6)}{(1-n)u_5u_9u_{11}}\right]x_2x_4\\ &\quad -\left[\frac{u_5u_{14}(u_9u_{15}-ne_5u_6)-(1-n)u_9u_{11}(u_5u_{15}\bar{x}_1+u_4u_{16}\bar{x}_2)}{(1-n)u_5u_9u_{11}}\right]x_4\\ &\quad -\left[\frac{(1-n)e_5u_8u_{11}-u_{10}(u_9u_{15}-ne_5u_6)}{(1-n)u_9u_{11}}\right]x_3\\ &\quad -\left[\frac{u_{13}(u_9u_{15}-ne_5u_6)}{(1-n)u_9u_{11}}+u_{17}\right]x_4x_5\\ &\quad -[u_{18}-u_{15}\bar{x}_1-u_{16}\bar{x}_2]x_5\end{aligned}$$

Now its easy to verify that $\frac{dL_3}{dt}$ is negative definite under the sufficient conditions (22a)-(22c). Hence the solution of system (2) will approach asymptotically to E_3 from any initial point satisfies the above condition and then the proof is complete. ■

Theorem (5): Assume that E_4 is locally asymptotically stable in R_+^5, then it is globally asymptotically stable provided that the following conditions hold:

$$\begin{aligned}&m(1-n)e_5u_5u_7u_{11}+(1-m)u_5u_9u_{12}u_{15}\\ &< (1-n)u_4u_9u_{11}u_{16}+n(1-m)e_5u_5u_6u_{12}\end{aligned} \tag{23a}$$

$$u_5u_{15}\hat{x}_1+u_1u_{16} < u_{16}^2\hat{x}_5 \tag{23b}$$

$$\left.\begin{aligned}&ne_5u_6+\frac{(1-n)u_9u_{11}(u_{15}\hat{x}_1+u_{17})}{u_{14}} < u_9u_{15}\\ &u_9u_{15} < \frac{(1-n)e_5u_{11}(u_9\hat{x}_5+u_8)}{u_{10}}+ne_5u_6\end{aligned}\right\} \tag{23c}$$

$$\frac{u_{18}}{u_{15}}\hat{x}_5 < (x_1-\hat{x}_1)^2 \tag{23d}$$

Proof: Consider the following function

$$\begin{aligned}L_4(x_1,\ldots,x_5) &= \hat{c}_1\left(x_1-\hat{x}_1-\hat{x}_1\ln\frac{x_1}{\hat{x}_1}\right)+\hat{c}_2x_2\\ &\quad +\hat{c}_3x_3+\hat{c}_4x_4+\hat{c}_5\left(x_5-\hat{x}_5-\hat{x}_5\ln\frac{x_5}{\hat{x}_5}\right)\end{aligned}$$

Where $\hat{c}_i, i=1,\ldots,5$ are positive constants to be determined. It is easy to see that $L_4(x_1,\ldots,x_5)\in C^1(R_+^5,R)$, and $L_4(\hat{x}_1,0,0,0,\hat{x}_5)=0$ while $L_4(x_1,\ldots,x_5)>0$ for all $(x_1,\ldots,x_5)\in R_+^5$ and $(x_1,\ldots,x_5)\neq(\hat{x}_1,0,0,0,\hat{x}_5)$. Further more by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned}\frac{dL_4}{dt} &= -\hat{c}_1(x_1-\hat{x}_1)^2 - \hat{c}_2u_1u_2x_2^2 - [\hat{c}_4u_{13}+\hat{c}_5u_{17}]x_4x_5\\ &\quad -(\hat{c}_1-\hat{c}_5u_{15})(x_1-\hat{x}_1)(x_5-\hat{x}_5) - (\hat{c}_2u_3+\hat{c}_1)x_1x_2\\ &\quad -[\hat{c}_5e_5\hat{x}_5+\hat{c}_3u_8-\hat{c}_4u_{10}]x_3 - [\hat{c}_3u_9-\hat{c}_5e_5]x_3x_5\\ &\quad -[\hat{c}_2u_{16}\hat{x}_5-\hat{c}_1\hat{x}_1-\hat{c}_2u_1]x_2 - [\hat{c}_2u_5-\hat{c}_5u_{16}]x_2x_5\\ &\quad -[\hat{c}_2u_4-\hat{c}_3mu_7-\hat{c}_4(1-m)u_{12}]x_2x_4\\ &\quad -[\hat{c}_1-\hat{c}_3nu_6-\hat{c}_4(1-n)u_{11}]x_1x_4 - \hat{c}_5u_{18}x_5\\ &\quad -[\hat{c}_4u_{14}-\hat{c}_1\hat{x}_1-\hat{c}_5u_{17}\hat{x}_5]x_4 + \hat{c}_5u_{18}\hat{x}_5\end{aligned}$$

Therefore by choosing the positive constants as below

$$\hat{c}_1 = 1,\quad \hat{c}_2 = \frac{u_{16}}{u_5u_{15}},\quad \hat{c}_3 = \frac{e_5}{u_9u_{15}},$$
$$\hat{c}_4 = \frac{u_9u_{15}-ne_5u_6}{(1-n)u_9u_{11}u_{15}},\quad \hat{c}_5 = \frac{1}{u_{15}}$$

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And then substituting these constants in the above equation, we get that

$$\frac{dL_4}{dt} = -\left(x_1 - \hat{x}_1\right)^2 - \frac{u_{16}}{u_5 u_{15}} u_1 u_2 x_2^2 - \frac{u_{18}}{u_{15}} x_5 + \frac{u_{18}}{u_{15}} \hat{x}_5$$
$$-\left[1 + \frac{u_3 u_{16}}{u_5 u_{15}}\right] x_1 x_2 - \left[\frac{u_{16}^2 \hat{x}_5 - u_5 u_{15} \hat{x}_1 - u_1 u_{16}}{u_5 u_{15}}\right] x_2$$
$$-\left[\frac{(1-n)u_9 u_{11}(u_4 u_{16} - m e_5 u_5 u_7) - (1-m) u_5 u_{12}(u_9 u_{15} - n e_5 u_6)}{(1-n) u_5 u_9 u_{11} u_{15}}\right] x_2 x_4$$
$$-\left[\frac{u_{14}(u_9 u_{15} - n e_5 u_6) - (1-n) u_9 u_{11}(u_{15}\hat{x}_1 + u_{17})}{(1-n) u_9 u_{11} u_{15}}\right] x_4$$
$$-\left[\frac{(1-n) e_5 u_{11}(u_9 \hat{x}_5 + u_8) - u_{10}(u_9 u_{15} - n e_5 u_6)}{(1-n) u_9 u_{11} u_{15}}\right] x_3$$
$$-\left[\frac{u_{13}(u_9 u_{15} - n u_6 e_5)}{(1-n) u_9 u_{11} u_{15}} + \frac{u_{17}}{u_{15}}\right] x_4 x_5$$

Now its easy to verify that $\frac{dL_4}{dt}$ is negative definite under the sufficient conditions (23a)-(23d). Hence the solution of system (2) will approach asymptotically to E_4 from any initial point satisfies the above condition and then the proof is complete. ■

Theorem (6): Assume that E_5 is locally asymptotically stable in R_+^5, then it is globally asymptotically stable provided that the following conditions hold:

$$m(1-n) e_5 u_5 u_7 u_{11} + (1-m) u_5 u_9 u_{12} u_{15} < (1-n) u_4 u_9 u_{11} u_{16} + n e_5 u_5 u_6 u_{12} (1-m) \quad ..(24a)$$

$$u_5 u_{15} + u_3 u_{16} \breve{x}_2 < u_5 u_{15} \breve{x}_5 \quad ..(24b)$$

$$\left.\begin{aligned} & n e_5 u_6 + \frac{(1-n) u_9 u_{11}(u_4 u_6 \breve{x}_2 + u_5 u_{17} \breve{x}_5)}{u_5 u_{14}} < u_9 u_{15} \\ & u_9 u_{15} < \frac{(1-n) e_5 u_{11}(u_9 \breve{x}_5 + u_8)}{u_{10}} + n e_5 u_6 \end{aligned}\right\} \quad \ldots(24c)$$

Proof: Consider the following functions

$$L_5(x_1, \ldots, x_5) = \breve{c}_1 x_1 + \breve{c}_2\left(x_2 - \breve{x}_2 - \breve{x}_2 L_n \frac{x_2}{\breve{x}_2}\right) + \breve{c}_3 x_3 + \breve{c}_4 x_4 + \breve{c}_5\left(x_5 - \breve{x}_5 - \breve{x}_5 L_n \frac{x_5}{\breve{x}_5}\right)$$

Where $\breve{c}_i, i = 1, \ldots, 5$ are positive constants to be determined. It is easy to see that $L_5(x_1, \ldots, x_5) \in C^1(R_+^5, R)$, and $L_5(0, \breve{x}_2, 0, 0, \breve{x}_5) = 0$ while $L_5(x_1, \ldots, x_5) > 0$ for all $(x_1, \ldots, x_5) \in R_+^5$ and $(x_1, \ldots, x_5) \neq (0, \breve{x}_2, 0, 0, \breve{x}_5)$. Further more by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\frac{dL_5}{dt} = -\breve{c}_1 x_1^2 - \left[\breve{c}_2 u_3 + \breve{c}_1\right] x_1 x_2 - \left[\breve{c}_1 - \breve{c}_5 u_{15}\right] x_1 x_5$$
$$- \breve{c}_2 u_1 u_2 (x_2 - \breve{x}_2)^2 - \left[\breve{c}_5 u_{15} \breve{x}_5 - \breve{c}_2 u_3 \breve{x}_2 - \breve{c}_1\right] x_1$$
$$- (\breve{c}_3 u_9 - \breve{c}_5 e_5) x_3 x_5 - (\breve{c}_4 u_{13} + \breve{c}_5 u_{17}) x_4 x_5$$
$$- \left[\breve{c}_2 u_4 - \breve{c}_3 m u_7 - \breve{c}_4 (1-m) u_{12}\right] x_2 x_4$$
$$- \left[\breve{c}_4 u_{14} - \breve{c}_2 u_4 \breve{x}_2 - \breve{c}_5 u_{17} \breve{x}_5\right] x_4$$
$$- (\breve{c}_2 u_5 - \breve{c}_5 u_{16})(x_2 - \breve{x}_2)(x_5 - \breve{x}_5)$$
$$- \left[\breve{c}_5 e_5 \breve{x}_5 + \breve{c}_3 u_8 - \breve{c}_4 u_{10}\right] x_3$$
$$- \left[\breve{c}_1 - \breve{c}_3 n u_6 - \breve{c}_4 (1-n) u_{11}\right] x_1 x_4$$

Now by choosing the positive constants as below

$$\breve{c}_1 = 1, \quad \breve{c}_2 = \frac{u_{16}}{u_5 u_{15}}, \quad \breve{c}_3 = \frac{e_5}{u_9 u_{15}},$$
$$\breve{c}_4 = \frac{u_9 u_{15} - n e_5 u_6}{(1-n) u_9 u_{11} u_{15}}, \quad \breve{c}_5 = \frac{1}{u_{15}}$$

And then substituting these constants in the above equation, we get that

$$\frac{dL_5}{dt} = -x_1^2 - \left[\frac{u_3 u_{16}}{u_5 u_{15}} + 1\right] x_1 x_2 - \frac{u_1 u_2 u_{16}}{u_5 u_{15}} (x_2 - \breve{x}_2)^2$$
$$-\left[\frac{(1-n) u_{11}(u_4 u_9 u_{16} - m e_5 u_7 u_5) - (1-m) u_5 u_{12}(u_9 u_{15} - n e_5 u_6)}{(1-n) u_5 u_9 u_{11} u_{15}}\right] x_2 x_4$$
$$-\left[\frac{(1-n) e_5 u_{11}(u_9 \breve{x}_5 + u_8) - u_{10}(u_9 u_{15} - n e_5 u_6)}{(1-n) u_9 u_{11} u_{15}}\right] x_3$$
$$-\left[\frac{u_5 u_{14}(u_9 u_{15} - n e_5 u_6) - (1-n) u_9 u_{11}(u_4 u_6 \breve{x}_2 + u_5 u_{17} \breve{x}_5)}{(1-n) u_5 u_9 u_{11} u_{15}}\right] x_4$$
$$-\left[\frac{u_5 u_{15} \breve{x}_5 - u_3 u_{16} \breve{x}_2 - u_5 u_{15}}{u_5 u_{15}}\right] x_1$$
$$-\left[\frac{u_{13}(u_9 u_{15} - n e_5 u_6)}{(1-n) u_9 u_{11} u_{15}} + \frac{u_{17}}{u_{15}}\right] x_4 x_5$$

Now its easy to verify that $\frac{dL_5}{dt}$ is negative definite under the suffecient conditions (24a)-(24c). Hence the solution of system (2) will approach asymptotically to E_5 from any initial point satisfies the above condition and then the proof is complete. ■

Theorem (7): Assume that E_6 is locally asymptotically stable in R_+^5, then it is globally asymptotically stable in the sub region of R_+^5 that satisfies the following conditions:

$$x_1 < \frac{u_{14}}{(1-n) u_{11}} \quad (25a)$$

$$\frac{u_5 u_{15} \bar{\bar{x}}_1 + u_5 u_{15} u_1}{u_1 u_2 u_{16}} < x_2 \quad (25b)$$

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$$\frac{u_9\bar{\bar{x}}_3+e_5}{u_9}<x_3 \quad (25c)$$

$$\sqrt{\frac{u_{15}\bar{\bar{x}}_1\bar{\bar{x}}_4}{u_{17}}}<x_4 \quad (25d)$$

$$\begin{aligned}&mu_5u_7u_{13}\bar{\bar{x}}_4x_3+(1-m)u_5u_{12}u_{17}x_4\\&<u_4u_{13}u_{16}\bar{\bar{x}}_4+mu_5u_7u_{13}\bar{\bar{x}}_3\bar{\bar{x}}_4+(1-m)u_5u_{12}u_{17}\bar{\bar{x}}_4\end{aligned} \quad ..(25e)$$

$$\left[nu_6\bar{\bar{x}}_4\right]^2<u_8u_{15} \quad (25f)$$

$$\left[u_{15}-(1-n)\frac{u_{11}u_{17}}{u_{13}}\right]^2<\frac{u_{15}\left[u_{14}u_{17}-(1-n)u_{11}u_{17}x_1\right]}{u_{13}\bar{\bar{x}}_4} \quad (25g)$$

$$\left[nu_6x_1+\frac{u_{10}u_{17}}{u_{13}\bar{\bar{x}}_4}\right]^2<\frac{u_8\left[u_{14}u_{17}-(1-n)u_{11}u_{17}x_1\right]}{u_{13}\bar{\bar{x}}_4} \quad ...(25h)$$

Proof: Consider the following function

$$\begin{aligned}L_6(x_1,\ldots,x_5)=\bar{\bar{c}}_1\left(x_1-\bar{\bar{x}}_1-\bar{\bar{x}}_1L_n\frac{x_1}{\bar{\bar{x}}_1}\right)+\bar{\bar{c}}_2x_2\\+\frac{\bar{\bar{c}}_3}{2}\left(x_3-\bar{\bar{x}}_3\right)^2+\frac{\bar{\bar{c}}_4}{2}\left(x_4-\bar{\bar{x}}_4\right)^2+\bar{\bar{c}}_5x_5\end{aligned}$$

Where $\bar{\bar{c}}_i, i=1,\ldots,5$ are positive constants to be determined. It is easy to see that $L_6(x_1,\ldots,x_5)\in C^1(R_+^5,R)$, and $L_6(\bar{\bar{x}}_1,0,\bar{\bar{x}}_3,\bar{\bar{x}}_4,0)=0$ while $L_6(x_1,\ldots,x_5)>0$ for all $(x_1,\ldots,x_5)\in R_+^5$ and $(x_1,\ldots,x_5)\neq(\bar{\bar{x}}_1,0,\bar{\bar{x}}_3,\bar{\bar{x}}_4,0)$. Further more by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned}\frac{dL_6}{dt}&=-\bar{\bar{c}}_1\left(x_1-\bar{\bar{x}}_1\right)^2-\bar{\bar{c}}_1x_1x_2+\bar{\bar{c}}_1\bar{\bar{x}}_1x_2-\bar{\bar{c}}_2u_1u_2x_2^2\\&+\left[\bar{\bar{c}}_3nu_6x_1+\bar{\bar{c}}_4u_{10}\right]\left(x_3-\bar{\bar{x}}_3\right)\left(x_4-\bar{\bar{x}}_4\right)+\bar{\bar{c}}_1\bar{\bar{x}}_1x_5\\&-\left[\bar{\bar{c}}_4u_{14}-\bar{\bar{c}}_4(1-n)u_{11}x_1\right]\left(x_4-\bar{\bar{x}}_4\right)^2-\bar{\bar{c}}_3u_9x_3^2x_5\\&-\left[\bar{\bar{c}}_1-\bar{\bar{c}}_4(1-n)u_{11}\bar{\bar{x}}_4\right]\left(x_1-\bar{\bar{x}}_1\right)\left(x_4-\bar{\bar{x}}_4\right)\\&+\bar{\bar{c}}_1u_1x_2-\bar{\bar{c}}_2u_3x_1x_2-\bar{\bar{c}}_3u_8\left(x_3-\bar{\bar{x}}_3\right)^2\\&-\left[\bar{\bar{c}}_2u_4+\bar{\bar{c}}_3mu_7\bar{\bar{x}}_3+\bar{\bar{c}}_4(1-m)u_{12}\bar{\bar{x}}_4\right]x_2x_4\\&-\left[\bar{\bar{c}}_2u_5-\bar{\bar{c}}_5u_{16}\right]x_2x_5-\left[\bar{\bar{c}}_1-\bar{\bar{c}}_5u_{15}\right]x_1x_5\\&+\bar{\bar{c}}_3nu_6\bar{\bar{x}}_4\left(x_1-\bar{\bar{x}}_1\right)\left(x_3-\bar{\bar{x}}_3\right)-\bar{\bar{c}}_4u_{13}x_4^2x_5\\&+\left[\bar{\bar{c}}_3u_9\bar{\bar{x}}_3+\bar{\bar{c}}_5e_5\right]x_3x_5+\bar{\bar{c}}_3mu_7x_2x_3x_4\\&+\bar{\bar{c}}_4(1-m)u_{12}x_2x_4^2-\left[\bar{\bar{c}}_5u_{17}-\bar{\bar{c}}_4u_{13}\bar{\bar{x}}_4\right]x_4x_5\end{aligned}$$

Now by choosing the positive constants as below

$$\bar{\bar{c}}_1=u_{15},\quad \bar{\bar{c}}_2=\frac{u_{16}}{u_5},\quad \bar{\bar{c}}_4=\frac{u_{17}}{u_{13}\bar{\bar{x}}_4},\quad \bar{\bar{c}}_3=\bar{\bar{c}}_5=1$$

and then substituting these constants in the above equation and using the conditions (25a),(25f)-(25h), we get that

$$\begin{aligned}\frac{dL_6}{dt}&\le-\left[\sqrt{\frac{u_{15}}{2}}\left(x_1-\bar{\bar{x}}_1\right)-\sqrt{\frac{u_{18}}{2}}\left(x_3-\bar{\bar{x}}_3\right)\right]^2\\&-\left[\sqrt{\frac{u_{15}}{2}}\left(x_1-\bar{\bar{x}}_1\right)+\sqrt{\frac{u_{17}\left(u_{14}-(1-n)u_{11}x_1\right)}{2u_{13}\bar{\bar{x}}_4}}\left(x_4-\bar{\bar{x}}_4\right)\right]^2\\&-\left[\sqrt{\frac{u_8}{2}}\left(x_3-\bar{\bar{x}}_3\right)-\sqrt{\frac{u_{17}\left(u_{14}-(1-n)u_{11}x_1\right)}{2u_{13}\bar{\bar{x}}_4}}\left(x_4-\bar{\bar{x}}_4\right)\right]^2\\&-\left[u_{15}+\frac{u_{16}}{u_5}u_3\right]x_1x_2-\left[\frac{u_{16}}{u_5}u_1u_2x_2-u_{15}\bar{\bar{x}}_1-u_{15}u_1\right]x_2\\&-\left[\frac{u_{17}}{\bar{\bar{x}}_4}x_4^2-u_{15}\bar{\bar{x}}_1\right]x_5-\left[u_9x_3-u_9\bar{\bar{x}}_3-e_5\right]x_3x_5\\&-\begin{bmatrix}\frac{u_{16}}{u_5}u_4+mu_7\bar{\bar{x}}_3+\frac{u_{17}}{u_{13}}(1-m)u_{12}\\-mu_7x_3-\frac{u_{17}}{u_{13}\bar{\bar{x}}_4}(1-m)u_{12}x_4\end{bmatrix}x_2x_4\end{aligned}$$

Now its easy to verify that $\frac{dL_6}{dt}$ is negative definite under the sufficient conditions (25b)-(25e). Hence the solution of system (2) will approach asymptotically to E_6 from any initial point satisfies the above condition and then the proof is complete. ■ Note that the stated sub region in the above theorem represents the basin of attraction of the equilibrium point E_6.

Theorem (8): Assume that E_7 is locally asymptotically stable in R_+^5, then it is globally asymptotically stable in the sub region of R_+^5 that satisfies the following conditions:

$$\frac{u_5u_{15}+u_3u_{16}\tilde{x}_2}{u_5u_{15}}<x_1 \quad (26a)$$

$$x_2<\frac{u_{14}}{(1-m)u_{12}} \quad (26b)$$

$$\frac{u_9\tilde{x}_3+e_5}{u_9}<x_3 \quad (26c)$$

$$\sqrt{\frac{u_{16}\tilde{x}_2\tilde{x}_4}{u_{17}}}<x_4 \quad (26d)$$

$$\left.\begin{aligned}&nu_6u_{13}\tilde{x}_4x_3+(1-n)u_{11}u_{17}x_4\\&<u_{13}u_{15}\tilde{x}_4+nu_6u_{13}\tilde{x}_3\tilde{x}_4+(1-n)u_{11}u_{17}\tilde{x}_4\end{aligned}\right\} \quad (26e)$$

$$\left[mu_7\tilde{x}_4\right]^2<\frac{u_1u_2u_8u_{16}}{u_5} \quad (26f)$$

$$\left.\begin{aligned}&\left[\frac{u_4u_{16}}{u_5}-\frac{u_{12}u_{17}}{u_{13}}(1-m)\right]^2\\&<\frac{u_1u_2u_{16}\left[u_{14}u_{17}-(1-m)u_{12}u_{17}x_2\right]}{u_5u_{13}\tilde{x}_4}\end{aligned}\right\} \quad (26g)$$

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$$\left.\begin{array}{l}\left[mu_7x_2+\dfrac{u_{10}u_{17}}{u_{13}\tilde{x}_4}\right]^2\\ <\dfrac{u_8\left[u_{14}u_{17}-(1-m)u_{12}u_{17}x_2\right]}{u_{13}\tilde{x}_4}\end{array}\right\} \qquad (26h)$$

Proof: Consider the following function

$$L_7(x_1,\ldots,x_5)=\tilde{c}_1x_1+\tilde{c}_2\left(x_2-\tilde{x}_2-\tilde{x}_2L_n\frac{x_2}{\tilde{x}_2}\right)+\frac{\tilde{c}_3}{2}(x_3-\tilde{x}_3)^2+\frac{\tilde{c}_4}{2}(x_4-\tilde{x}_4)^2+\tilde{c}_5x_5$$

where $\tilde{c}_i, i=1,\ldots,5$ are positive constants to be determined. It is easy to see that $L_7(x_1,\ldots,x_5)\in C^1(R_+^5,R)$, and $L_7(0,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4,0)=0$ while $L_7(x_1,\ldots,x_5)>0$ for all $(x_1,\ldots,x_5)\in R_+^5$ and $(x_1,\ldots,x_5)\neq(0,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4,0)$. Further more by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned}\frac{dL_7}{dt}=&-[\tilde{c}_1x_1-\tilde{c}_1-\tilde{c}_2u_3\tilde{x}_2]x_1-[\tilde{c}_1+\tilde{c}_2u_3]x_1x_2\\&+\tilde{c}_3mu_7x_2(x_3-\tilde{x}_3)(x_4-\tilde{x}_4)-[\tilde{c}_5u_{17}-\tilde{c}_4u_{13}\tilde{x}_4]x_4x_5\\&-\left[\tilde{c}_4u_{13}x_4^2-\tilde{c}_2u_5\tilde{x}_2\right]x_5+\tilde{c}_4(1-m)u_{12}x_2(x_4-\tilde{x}_4)^2\\&+\tilde{c}_3mu_7\tilde{x}_4(x_2-\tilde{x}_2)(x_3-\tilde{x}_3)-[\tilde{c}_1-\tilde{c}_5u_{15}]x_1x_5\\&-\tilde{c}_2u_1u_2(x_2-\tilde{x}_2)^2-\tilde{c}_2u_4(x_2-\tilde{x}_2)(x_4-\tilde{x}_4)\\&+\tilde{c}_4u_{10}(x_3-\tilde{x}_3)(x_4-\tilde{x}_4)-\tilde{c}_3u_8(x_3-\tilde{x}_3)^2\\&-[\tilde{c}_2u_5-\tilde{c}_5u_{16}]x_2x_5-\tilde{c}_4u_{14}(x_4-\tilde{x}_4)^2\\&-\begin{bmatrix}\tilde{c}_1+\tilde{c}_3nu_6\tilde{x}_3+\tilde{c}_4(1-n)u_{11}\tilde{x}_4\\-\tilde{c}_4(1-n)u_{11}x_4-\tilde{c}_3nu_6x_3\end{bmatrix}x_1x_4\\&-[\tilde{c}_3u_9x_3-\tilde{c}_3u_9\tilde{x}_3-\tilde{c}_5e_5]x_3x_5\\&+\tilde{c}_4(1-m)u_{12}\tilde{x}_4(x_2-\tilde{x}_2)(x_4-\tilde{x}_4)\end{aligned}$$

So by choosing the positive constants as below

$$\tilde{c}_1=u_{15},\ \tilde{c}_2=\frac{u_{16}}{u_5},\quad \tilde{c}_4=\frac{u_{17}}{u_{13}\tilde{x}_4},\quad \tilde{c}_3=\tilde{c}_5=1$$

and then substituting these constants in the above equation and using the conditions (26b),(26f)-(26h), we get that

$$\begin{aligned}\frac{dL_7}{dt}\le&-\left[\sqrt{\frac{u_1u_2u_{16}}{2u_5}}(x_2-\tilde{x}_2)-\sqrt{\frac{u_8}{2}}(x_3-\tilde{x}_3)\right]^2\\&-\left[\sqrt{\frac{u_1u_2u_{16}}{2u_5}}(x_2-\tilde{x}_2)+\right.\\&\qquad\left.\sqrt{\frac{u_{17}(u_{14}-(1-m)u_{12}x_2)}{2u_{13}\tilde{x}_4}}(x_4-\tilde{x}_4)\right]^2\\&-\left[\sqrt{\frac{u_8}{2}}(x_3-\tilde{x}_3)-\sqrt{\frac{u_{17}(u_{14}-(1-m)u_{12}x_2)}{2u_{13}\tilde{x}_4}}(x_4-\tilde{x}_4)\right]^2\\&-\left[u_{15}x_1-u_{15}-\frac{u_3u_{16}}{u_5}\tilde{x}_2\right]x_1-\left[u_{15}+\frac{u_3u_{16}}{u_5}\right]x_1x_2\\&-\left[\frac{u_{17}}{\tilde{x}_4}x_4^2-u_{16}\tilde{x}_2\right]x_5-[u_9x_3-u_9\tilde{x}_3-e_5]x_3x_5\\&-\left[u_{15}+nu_6\tilde{x}_3+\frac{u_{11}u_{17}}{u_{13}}(1-n)-\right.\\&\qquad\left.\frac{u_{17}}{u_{13}\tilde{x}_4}(1-n)u_{11}x_4-nu_6x_3\right]x_1x_4\end{aligned}$$

Now its easy to verify that $\frac{dL_7}{dt}$ is negative definite under the sufficient conditions (26a), (26c)-(25e). Hence the solution of system (2) will approach asymptotically to E_7 from any initial point satisfies the above condition and then the proof is complete. ■

Theorem (9): Assume that the third three species equilibrium point E_8 exists, then it is a globally asymptotically stable in R_+^5, if the following conditions hold

$$\left.\begin{array}{l}m(1-n)e_5u_5u_7u_{11}+(1-m)u_5u_9u_{12}u_{15}\\<(1-n)u_4u_9u_{11}u_{16}+ne_5u_5u_6u_{12}(1-m)\end{array}\right\} \qquad (27a)$$

$$\left.\begin{array}{l}\dfrac{\left[ne_5u_6u_{14}+(1-n)u_9u_{11}u_{15}\tilde{\tilde{x}}_1\right]}{u_{14}}+\dfrac{(1-n)u_9u_{11}\left[u_4\tilde{\tilde{x}}_2+u_5u_{17}\tilde{\tilde{x}}_5\right]}{u_5u_{14}}\\<u_9u_{15}<ne_5u_6+\dfrac{(1-n)e_5u_{11}\left[u_8+u_9\tilde{\tilde{x}}_5\right]}{u_{10}}\end{array}\right\} \qquad (27b)$$

$$\left[\frac{u_5u_{15}+u_3u_{16}}{u_5}\right]^2<4\frac{u_1u_2u_{15}u_{16}}{u_5} \qquad (27c)$$

Proof: Consider the following function

$$L_8(x_1,\ldots,x_5)=\tilde{\tilde{c}}_1\left(x_1-\tilde{\tilde{x}}_1-\tilde{\tilde{x}}_1\ln\frac{x_1}{\tilde{\tilde{x}}_1}\right)+\tilde{\tilde{c}}_3x_3+\tilde{\tilde{c}}_4x_4+\tilde{\tilde{c}}_2\left(x_2-\tilde{\tilde{x}}_2-\tilde{\tilde{x}}_2\ln\frac{x_2}{\tilde{\tilde{x}}_2}\right)+\tilde{\tilde{c}}_5\left(x_5-\tilde{\tilde{x}}_5-\tilde{\tilde{x}}_5\ln\frac{x_5}{\tilde{\tilde{x}}_5}\right)$$

where $\tilde{\tilde{c}}_i, i=1,\ldots,5$ are positive constants to be determined. It is easy to see that $L_8(x_1,\ldots,x_5)\in C^1(R_+^5,R)$, and $L_8(\tilde{\tilde{x}}_1,\tilde{\tilde{x}}_2,0,0,\tilde{\tilde{x}}_5)=0$ while $L_8(x_1,\ldots,x_5)>0$ for all $(x_1,\ldots,x_5)\in R_+^5$ and $(x_1,\ldots,x_5)\neq(\tilde{\tilde{x}}_1,\tilde{\tilde{x}}_2,0,0,\tilde{\tilde{x}}_5)$. Further more by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned}\frac{dL_8}{dt}=&-\tilde{\tilde{c}}_1\left(x_1-\tilde{\tilde{x}}_1\right)^2-\left[\tilde{\tilde{c}}_1+\tilde{\tilde{c}}_2u_3\right]\left(x_1-\tilde{\tilde{x}}_1\right)\left(x_2-\tilde{\tilde{x}}_2\right)\\&-\left[\tilde{\tilde{c}}_1-\tilde{\tilde{c}}_5u_{15}\right]\left(x_1-\tilde{\tilde{x}}_1\right)\left(x_5-\tilde{\tilde{x}}_5\right)-\tilde{\tilde{c}}_2u_1u_2\left(x_2-\tilde{\tilde{x}}_2\right)^2\\&-\left[\tilde{\tilde{c}}_3u_9-\tilde{\tilde{c}}_5e_5\right]x_3x_5-\left[\tilde{\tilde{c}}_4u_{13}+\tilde{\tilde{c}}_5u_{17}\right]x_4x_5\\&-\left[\tilde{\tilde{c}}_4u_{14}-\tilde{\tilde{c}}_1\tilde{\tilde{x}}_1-\tilde{\tilde{c}}_2u_4\tilde{\tilde{x}}_2-\tilde{\tilde{c}}_5u_{17}\tilde{\tilde{x}}_5\right]x_4\\&-\left[\tilde{\tilde{c}}_2u_4-\tilde{\tilde{c}}_3mu_7-\tilde{\tilde{c}}_4(1-m)u_{12}\right]x_2x_4\\&-\left[\tilde{\tilde{c}}_2u_5-\tilde{\tilde{c}}_5u_{16}\right]\left(x_2-\tilde{\tilde{x}}_2\right)\left(x_5-\tilde{\tilde{x}}_5\right)\\&-\left[\tilde{\tilde{c}}_1-\tilde{\tilde{c}}_3nu_6-\tilde{\tilde{c}}_4(1-n)u_{11}\right]x_1x_4\\&-\left[\tilde{\tilde{c}}_3u_8-\tilde{\tilde{c}}_4u_{10}+\tilde{\tilde{c}}_5e_5\tilde{\tilde{x}}_5\right]x_3\end{aligned}$$

So by choosing the positive constants as below

$$\tilde{\tilde{c}}_1=u_{15},\quad \tilde{\tilde{c}}_2=\frac{u_{16}}{u_5},\quad \tilde{\tilde{c}}_3=\frac{e_5}{u_9},$$

$$\tilde{\tilde{c}}_4=\frac{u_9u_{15}-nu_6e_5}{(1-n)u_9u_{11}},\quad \tilde{\tilde{c}}_5=1$$

Then substituting these constants in the above equation and using the condition (27c), we get that

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$$\frac{dL_8}{dt} \le -\left[\sqrt{u_{15}}\left(x_1-\tilde{\tilde{x}}_1\right)+\sqrt{\frac{u_1u_2u_{16}}{u_5}}\left(x_2-\tilde{\tilde{x}}_2\right)\right]^2$$
$$-\left[\frac{(1-n)u_{11}(u_4u_9u_{16}-me_5u_5u_7)}{(1-n)u_5u_9u_{11}}-\frac{(1-m)u_5u_{12}(u_9u_{15}-nu_6e_5)}{(1-n)u_5u_9u_{11}}\right]x_2x_4$$
$$-\left[\frac{(1-n)e_5u_{11}\left(u_8+u_9\tilde{\tilde{x}}_5\right)-u_{10}(u_9u_{15}-nu_6e_5)}{(1-n)u_9u_{11}}\right]x_3$$
$$-\left[\frac{u_{13}(u_9u_{15}-ne_5u_6)}{(1-n)u_9u_{11}}+u_{17}\right]x_4x_5$$
$$-\left[\frac{u_5u_{14}(u_9u_{15}-ne_5u_6)u_5u_{14}}{(1-n)u_5u_9u_{11}}-\frac{(1-n)u_5u_9u_{11}\left(u_{15}\tilde{\tilde{x}}_1+u_{17}\tilde{\tilde{x}}_5\right)-(1-n)u_4u_9u_{11}\tilde{\tilde{x}}_2}{(1-n)u_5u_9u_{11}}\right]x_4$$

Now its easy to verify that $\frac{dL_8}{dt}$ is negative definite under the sufficient conditions (27a)-(27b). Hence the solution of system (2) will approach asymptotically to E_8 from any initial point satisfies the above condition and then the proof is complete. ■

Theorem (10): Assume that the top predator free equilibrium point E_9 exists, then it is a globally asymptotically stable in the sub region of R_+^5 that satisfies the following conditions:

$$\frac{u_9\breve{\breve{x}}_3+e_5}{u_9}<x_3 \quad (28a)$$

$$u_{15}\breve{\breve{x}}_1+u_{16}\breve{\breve{x}}_2<u_{18} \quad (28b)$$

$$(1-n)u_{11}x_1+(1-m)u_{12}x_2<u_{14} \quad (28c)$$

$$d_{12}{}^2<\frac{4}{9}\frac{u_1u_2u_5u_{16}}{u_5} \quad (28d)$$

$$d_{14}{}^2<\frac{4}{9}u_{15}d_{44} \quad (28e)$$

$$d_{24}{}^2<\frac{4}{9}\frac{u_1u_2u_{16}}{u_5}d_{44} \quad (28f)$$

$$d_{34}{}^2<\frac{4}{9}u_8d_{44} \quad (28g)$$

$$\left(nu_6\breve{\breve{x}}_4\right)^2<\frac{4}{9}u_8u_{15} \quad (28h)$$

$$\left(mu_7\breve{\breve{x}}_4\right)^2<\frac{4}{9}u_1u_2u_8u_{16} \quad (28i)$$

where

$$d_{12}=\frac{u_5u_{15}+u_3u_{16}}{u_5}, \qquad d_{24}=\frac{u_{13}u_{15}-(1-n)u_{11}u_{17}}{u_{13}},$$

$$d_{24}=\frac{u_4u_{13}u_{16}-(1-m)u_5u_{12}u_{17}}{u_5u_{13}},$$

$$d_{34}=\frac{nu_6u_{13}\breve{\breve{x}}_4x_1+mu_7u_{13}\breve{\breve{x}}_4x_2+u_{10}u_{17}}{u_{13}\breve{\breve{x}}_4} \quad \text{and}$$

$$d_{44}=u_{17}\frac{u_{14}-(1-n)u_{11}x_1-(1-m)u_{12}x_2}{u_{13}\breve{\breve{x}}_4}$$

Proof: Consider the following function

$$L_9(x_1,\ldots,x_5)=\breve{\breve{c}}_1\left(x_1-\breve{\breve{x}}_1-\breve{\breve{x}}_1\ln\frac{x_1}{\breve{\breve{x}}_1}\right)+\frac{\breve{\breve{c}}_3}{2}\left(x_3-\breve{\breve{x}}_3\right)^2$$
$$+\breve{\breve{c}}_2\left(x_2-\breve{\breve{x}}_2-\breve{\breve{x}}_2\ln\frac{x_2}{\breve{\breve{x}}_2}\right)+\frac{\breve{\breve{c}}_4}{2}\left(x_4-\breve{\breve{x}}_4\right)^2+\breve{\breve{c}}_5x_5$$

where $\breve{\breve{c}}_i, i=1,\ldots,5$ are positive constants to be determined. It is easy to see that $L_9(x_1,\ldots,x_5)\in C^1(R_+^5,R)$, and $L_9(\breve{\breve{x}}_1,\breve{\breve{x}}_2,\breve{\breve{x}}_3,\breve{\breve{x}}_4,0)=0$ while $L_9(x_1,\ldots,x_5)>0$ for all $(x_1,\ldots,x_5)\in R_+^5$ and $(x_1,\ldots,x_5)\neq(\breve{\breve{x}}_1,\breve{\breve{x}}_2,\breve{\breve{x}}_3,\breve{\breve{x}}_4,0)$. Further more by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\frac{dL_9}{dt}=-\breve{\breve{c}}_1\left(x_1-\breve{\breve{x}}_1\right)^2-\left[\breve{\breve{c}}_1+\breve{\breve{c}}_2u_3\right]\left(x_1-\breve{\breve{x}}_1\right)\left(x_2-\breve{\breve{x}}_2\right)$$
$$-\left[\breve{\breve{c}}_4u_{14}-\breve{\breve{c}}_4(1-n)u_{11}x_1-\breve{\breve{c}}_4(1-m)u_{12}x_2\right]\left(x_4-\breve{\breve{x}}_4\right)^2$$
$$-\left[\breve{\breve{c}}_3u_9x_3-\breve{\breve{c}}_3u_9\breve{\breve{x}}_3-\breve{\breve{c}}_5e_5\right]x_3x_5-\left[\breve{\breve{c}}_1-\breve{\breve{c}}_5u_{15}\right]x_1x_5$$
$$+\left[\breve{\breve{c}}_3nu_6x_1+\breve{\breve{c}}_3mu_7x_2+\breve{\breve{c}}_4u_{10}\right]\left(x_3-\breve{\breve{x}}_3\right)\left(x_4-\breve{\breve{x}}_4\right)$$
$$-\left[\breve{\breve{c}}_5u_{18}-\breve{\breve{c}}_1\breve{\breve{x}}_1-\breve{\breve{c}}_2u_5\breve{\breve{x}}_2\right]x_5-\left[\breve{\breve{c}}_2u_5-\breve{\breve{c}}_5u_{16}\right]x_2x_5$$
$$+\breve{\breve{c}}_3nu_6\breve{\breve{x}}_4\left(x_1-\breve{\breve{x}}_1\right)\left(x_3-\breve{\breve{x}}_3\right)-\breve{\breve{c}}_2u_1u_2\left(x_2-\breve{\breve{x}}_2\right)^2$$
$$+\breve{\breve{c}}_3mu_7\breve{\breve{x}}_4\left(x_2-\breve{\breve{x}}_2\right)\left(x_3-\breve{\breve{x}}_3\right)-\breve{\breve{c}}_3u_8\left(x_3-\breve{\breve{x}}_3\right)^2$$
$$-\left[\breve{\breve{c}}_2u_4-\breve{\breve{c}}_4(1-m)u_{12}\breve{\breve{x}}_4\right]\left(x_2-\breve{\breve{x}}_2\right)\left(x_4-\breve{\breve{x}}_4\right)$$
$$-\left[\breve{\breve{c}}_1-\breve{\breve{c}}_4(1-n)u_{11}\breve{\breve{x}}_4\right]\left(x_1-\breve{\breve{x}}_1\right)\left(x_4-\breve{\breve{x}}_4\right)$$
$$-\left[\breve{\breve{c}}_5u_{17}-\breve{\breve{c}}_4u_{13}\breve{\breve{x}}_4\right]x_4x_5-\breve{\breve{c}}_4u_{13}x_4^2x_5$$

So by choosing the positive constants as below

$$\breve{\breve{c}}_1=u_{15},\quad \breve{\breve{c}}_2=\frac{u_{16}}{u_5},\quad \breve{\breve{c}}_4=\frac{u_{17}}{u_{13}\breve{\breve{x}}_4},\quad \breve{\breve{c}}_3=\breve{\breve{c}}_5=1$$

Then substituting these constants in the above equation and using the condition (28c)-(28i), we get that

$$\frac{dL_9}{dt}\le-\left[\sqrt{\frac{u_{15}}{3}}\left(x_1-\breve{\breve{x}}_1\right)+\sqrt{\frac{u_1u_2u_{16}}{3u_5}}\left(x_2-\breve{\breve{x}}_2\right)\right]^2$$
$$-\left[\sqrt{\frac{u_1u_2u_{16}}{3u_5}}\left(x_2-\breve{\breve{x}}_2\right)+\sqrt{\frac{d_{44}}{3}}\left(x_4-\breve{\breve{x}}_4\right)\right]^2$$
$$-\left[\sqrt{\frac{u_1u_2u_{16}}{3u_5}}\left(x_2-\breve{\breve{x}}_2\right)-\sqrt{\frac{u_8}{3}}\left(x_3-\breve{\breve{x}}_3\right)\right]^2$$
$$-\left[\sqrt{\frac{u_{15}}{3}}\left(x_1-\breve{\breve{x}}_1\right)+\sqrt{\frac{d_{44}}{3}}\left(x_4-\breve{\breve{x}}_4\right)\right]^2$$
$$-\left[\sqrt{\frac{u_8}{3}}\left(x_3-\breve{\breve{x}}_3\right)-\sqrt{\frac{d_{44}}{3}}\left(x_4-\breve{\breve{x}}_4\right)\right]^2$$
$$-\left[\sqrt{\frac{u_{15}}{3}}\left(x_1-\breve{\breve{x}}_1\right)-\sqrt{\frac{u_8}{3}}\left(x_3-\breve{\breve{x}}_3\right)\right]^2$$
$$-\left[u_{18}-u_{15}\breve{\breve{x}}_1-u_{16}\breve{\breve{x}}_2\right]x_5-\frac{u_{17}}{\breve{\breve{x}}_4}x_4^2x_5$$
$$-\left[u_9x_3-u_9\breve{\breve{x}}_3-e_5\right]x_3x_5$$

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Now its easy to verify that $\frac{dL_9}{dt}$ is negative definite under the sufficient conditions (28a)-(28b). Hence the solution of system (2) will approach asymptotically to E_9 from any initial point satisfies the above condition and then the proof is complete. ■

Theorem (11): Assume that the positive equilibrium E_{10} exists, then it is a globally asymptotically stable in the sub region of R_+^5 that satisfies the following conditions:

$$\frac{e_5}{u_9} < x_3 \tag{29a}$$

$$(1-n)u_{11}x_1 + (1-m)u_{12}x_2 < u_{13}x_5^* \tag{29b}$$

$$\left.\begin{array}{c} x_3 > x_3^*, x_4 > x_4^* \text{ with } x_5 > x_5^* \\ \text{OR} \\ x_3 < x_3^*, x_4 < x_4^* \text{ with } x_5 < x_5^* \end{array}\right\} \tag{29c}$$

$$q_{12}{}^2 < \frac{4}{9}\frac{u_1u_2u_{15}u_{16}}{u_5} \tag{29d}$$

$$q_{14}{}^2 < \frac{4}{9}u_{15}q_{44} \tag{29e}$$

$$q_{24}{}^2 < \frac{4}{9}\frac{u_1u_2u_{16}}{u_5}q_{44} \tag{29f}$$

$$q_{34}{}^2 < \frac{4}{9}\left(u_8 + u_9x_5^*\right)q_{44} \tag{29g}$$

$$\left(nu_6x_4^*\right)^2 < \frac{4}{9}u_{15}\left(u_8 + u_9x_5^*\right) \tag{29h}$$

$$\left(mu_7x_4^*\right)^2 < \frac{4}{9}\frac{u_1u_2u_{16}\left(u_8 + u_9x_5^*\right)}{u_5} \tag{29i}$$

where $q_{12} = u_{15} + \frac{u_{16}u_3}{u_5}$, $q_{14} = u_{15} - (1-n)u_{11}x_4^*$,

$q_{24} = \frac{u_4u_{16}}{u_5} - (1-m)u_{12}x_4^*$, $q_{34} = nu_6x_1 + mu_7x_2 + u_{10}$

and $q_{44} = u_{13}x_5^* - (1-n)u_{11}x_1 - (1-m)u_{12}x_2$.

Proof: Consider the following function

$$\begin{aligned} L_{10}(x_1,\dots,x_5) &= c_1^*\left(x_1 - x_1^* - x_1^*\ln\frac{x_1}{x_1^*}\right) \\ &+ c_2^*\left(x_2 - x_2^* - x_2^*\ln\frac{x_2}{x_2^*}\right) + \frac{c_4^*}{2}\left(x_4 - x_4^*\right)^2 \\ &+ c_5^*\left(x_5 - x_5^* - x_5^*\ln\frac{x_5}{x_5^*}\right) + \frac{c_3^*}{2}\left(x_3 - x_3^*\right)^2 \end{aligned}$$

where $c_i^*, i = 1,\dots,5$ are positive constants to be determined. It is easy to see that $L_{10}(x_1,\dots,x_5) \in C^1\left(R_+^5, R\right)$ and $L_{10}\left(x_1^*,x_2^*,x_3^*,x_4^*,x_5^*\right) = 0$ while $L_{10}(x_1,\dots,x_5) > 0$ for all $(x_1,\dots,x_5) \in R_+^5$ and $(x_1,\dots,x_5) \ne \left(x_1^*,x_2^*,x_3^*,x_4^*,x_5^*\right)$. Further more by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned} \frac{dL_{10}}{dt} &= -c_1^*\left(x_1 - x_1^*\right)^2 - \left[c_1^* + c_2^*u_3\right]\left(x_1 - x_1^*\right)\left(x_2 - x_2^*\right) \\ &+ \left[c_3^*nu_6x_1 + c_3^*mu_7x_2 + c_4^*u_{10}\right]\left(x_3 - x_3^*\right)\left(x_4 - x_4^*\right) \\ &- \left[c_3^*u_8 + c_3^*u_9x_5^*\right](x_3 - x_3^*)^2 - c_2^*u_1u_2\left(x_2 - x_2^*\right)^2 \\ &- \left[c_2^*u_4 - c_4^*(1-m)u_{12}x_4^*\right]\left(x_2 - x_2^*\right)\left(x_4 - x_4^*\right) \\ &- \left[c_1^* - c_4^*(1-n)u_{11}x_4^*\right]\left(x_1 - x_1^*\right)\left(x_4 - x_4^*\right) \\ &- \left[\begin{array}{l} c_4^*u_{13}x_5^* - c_4^*(1-n)u_{11}x_1 \\ - c_4^*(1-m)u_{12}x_2 \end{array}\right](x_4 - x_4^*)^2 \\ &- \left[c_4^*u_{13}x_4 + c_5^*u_{17}\right](x_4 - x_4^*)(x_5 - x_5^*) \\ &- \left[c_2^*u_5 - c_5^*u_{16}\right]\left(x_2 - x_2^*\right)\left(x_5 - x_5^*\right) \\ &- \left[c_3^*u_9x_3 - c_5^*e_5\right]\left(x_3 - x_3^*\right)\left(x_5 - x_5^*\right) \\ &- \left[c_1^* - c_5^*u_{15}\right]\left(x_1 - x_1^*\right)\left(x_5 - x_5^*\right) \\ &+ c_3^*nu_6x_4^*(x_1 - x_1^*)(x_3 - x_3^*) \\ &+ c_3^*mu_7x_4^*(x_2 - x_2^*)(x_3 - x_3^*) \end{aligned}$$

By choosing the positive constants as below

$$c_1^* = u_{15},\quad c_2^* = \frac{u_{16}}{u_5},\quad c_3^* = c_4^* = c_5^* = 1$$

Then substituting these constants in the above equation and using the condition (29b) and (29d)-(29i), we get that

$$\begin{aligned} \frac{dL_{10}}{dt} < &-\left[\sqrt{\frac{u_{15}}{3}}\left(x_1 - x_1^*\right) + \sqrt{\frac{u_1u_2u_{16}}{3u_5}}\left(x_2 - x_2^*\right)\right]^2 \\ &-\left[\sqrt{\frac{u_1u_2u_{16}}{3u_5}}\left(x_2 - x_2^*\right) - \sqrt{\frac{u_8 + u_9x_5^*}{3}}\left(x_3 - x_3^*\right)\right]^2 \\ &-\left[\sqrt{\frac{u_{15}}{3}}\left(x_1 - x_1^*\right) - \sqrt{\frac{u_8 + u_9x_5^*}{3}}\left(x_3 - x_3^*\right)\right]^2 \\ &-\left[\sqrt{\frac{u_8 + u_9x_5^*}{3}}\left(x_3 - x_3^*\right) - \sqrt{\frac{q_{44}}{3}}\left(x_4 - x_4^*\right)\right]^2 \\ &-\left[\sqrt{\frac{u_1u_2u_{16}}{3u_5}}\left(x_2 - x_2^*\right) + \sqrt{\frac{q_{44}}{3}}\left(x_4 - x_4^*\right)\right]^2 \\ &-\left[\sqrt{\frac{u_{15}}{3}}\left(x_1 - x_1^*\right) + \sqrt{\frac{q_{44}}{3}}\left(x_4 - x_4^*\right)\right]^2 \\ &-\left[u_{13}x_4 + u_{17}\right]\left(x_4 - x_4^*\right)\left(x_5 - x_5^*\right) \\ &-\left[u_9x_3 - e_5\right]\left(x_3 - x_3^*\right)\left(x_5 - x_5^*\right) \end{aligned}$$

Now its easy to verify that $\frac{dL_{10}}{dt}$ is negative definite under the sufficient conditions (29a) and (29c). Hence the solution of system (2) will approach asymptotically to E_{10} from any initial point satisfies the above condition and then the proof is complete. ■

5. Numerical Simulation:

In this section, the dynamics behavior of system (2) is studied numerically. The objectives of this study are

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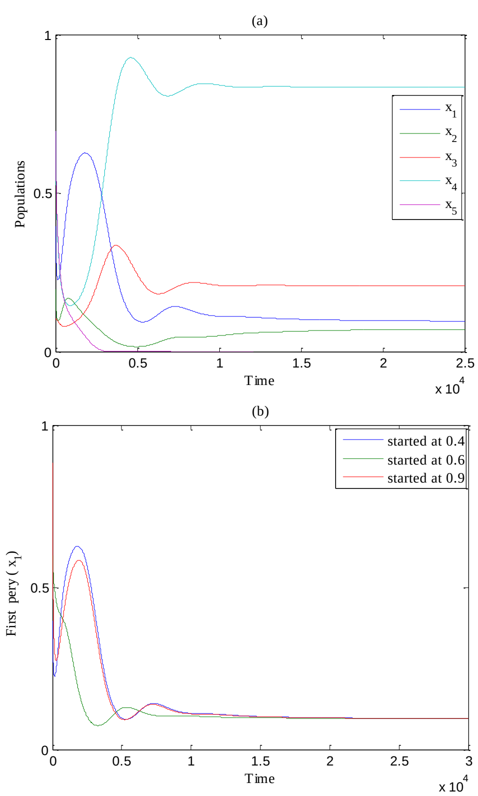
confirming our obtained analytical results and understand the effects of some parameters on the dynamics of system (2). Consequently, the system (2) is solved numerically for different sets of initial conditions and for different sets of parameters. Recall that system (2) contains two enter-specific competitions interactions, the first one between the two preys at the first level while the second one between the mature predator in the second level and the top predator at the third level. Although, the competitive exclusion principle states that "two species that compete for the exactly same resources cannot stably coexist"; the existence of predator makes the coexistence of all species possible. Therefore we can't find hypothetical set of data satisfy the coexistence of all the species together, rather than that we found the set of data that satisfy the coexistence for four populations of them as given below. Moreover since we presents the conditions that make the system has an asymptotically stable positive equilibrium point analytically, hence still there is possibility to have such a data. It is observed that, for the following set of hypothetical parameters values, system (2) has an asymptotically stable top predator free equilibrium point E_9 as shown in Fig. (1).

$$\begin{aligned}&u_1=1.2,\ u_2=1.5,\ u_3=1.19,\ u_4=1.15,\ u_5=1.15,\\&u_6=0.3,\ m=n=0.5,\ u_7=0.3, u_8=0.1,\ u_9=0.44,\\&u_{10}=0.1, u_{11}=0.3,\ u_{12}=0.3,\ u_{13}=1,\ u_{14}=0.05,\\&u_{15}=0.15, u_{16}=0.3,\ u_{17}=0.9,\ u_{18}=0.1,\ e_5=0.25\end{aligned} \qquad \ldots\ldots(30)$$



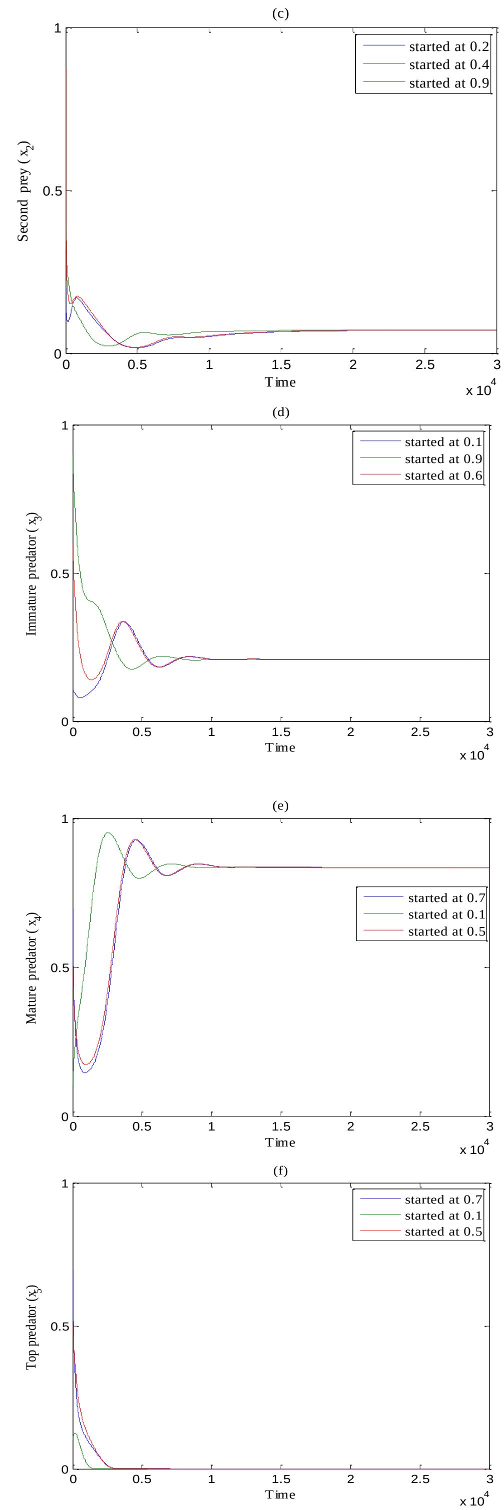


Fig. 1: *Time series of the solution of system (2) for data given by (30). (a) The trajectories of all species starting at* $(0.9,0.9,0.6,0.5,0.5)$ *. (b) The trajectories of* x_1 *- species starting from three different initial points. (c) The trajectories of* x_2 *- species starting from three different initial points. (d) The trajectories of* x_3 *- species starting from three different initial points. (e) The trajectories of* x_4 *- species starting from three different initial points. (f) The trajectories of* x_5 *- species starting from three different initial points.*

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However, for the data given by Eq. (30) with initial point $(0.3,0.5,0.2,0.3,0.6)$ that different from those used in Fig. (1), the trajectory of system (2) approaches asymptotically to third three species equilibrium point E_8 as drawn in figure (2).

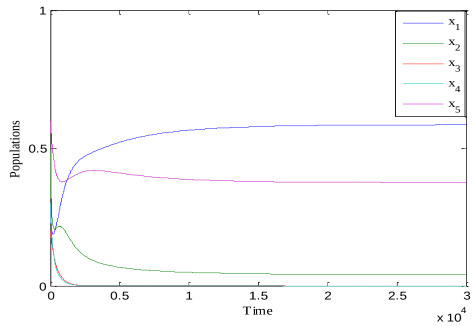
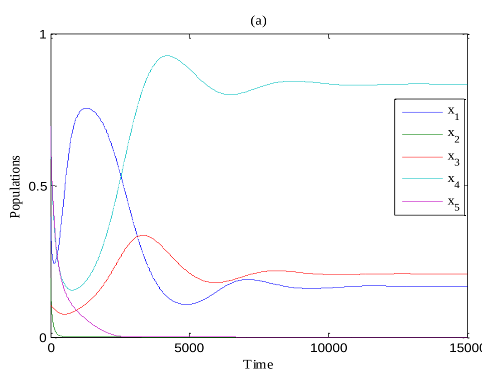


***Fig. 2:** Time series of the solution of system (2), for the data given by (30) with initial point* $(0.3,0.5,0.2,0.3,0.6)$ *,that approaches asymptotically to* $E_8 = (0.58, 0.04, 0, 0, 0.37)$

Obviously Fig. (1) and Fig. (2), show clearly the existence of sub region of global stability (basin of attraction) for each equilibrium points of system (2). This confirms our obtained analytical results present in the previous section. Indeed the initial points used in Fig. (1) satisfy the conditions given in theorem (10), while the initial point used in Fig. (2) satisfies the conditions in theorem (9). Note that in order to discuss the effect of the parameters values of system (2) on the dynamical behavior of system (2), the system is solved numerically for the data given in Eq. (30) with varying one parameter each time. It is observed that, for the above hypothetical data, the parameters values $u_i, i = 5,6,7,10,12,18$, m and n don't have qualitative effect on the dynamical behavior of system (2) and the system still approaches to a top predator free equilibrium point E_9, rather than that they have quantitative effect on the position of E_9. Now by varying the parameter u_1 keeping the rest of parameters values as in Eq. (30), it observed that for $u_1 \leq 1.14$ system (2) approaches asymptotically to $E_6 = (\bar{\bar{x}}_1, 0, \bar{\bar{x}}_3, \bar{\bar{x}}_4, 0)$, while for $1.32 \leq u_1 \leq 1.75$ the solution of system (2) approaches asymptotically to $E_7 = (0, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, 0)$, Further for $u_1 > 1.75$ the solution approaches asymptotically to $E_5 = (0, \breve{x}_2, 0, 0, \breve{x}_5)$ as shown in the typical figure given by Fig. (3).



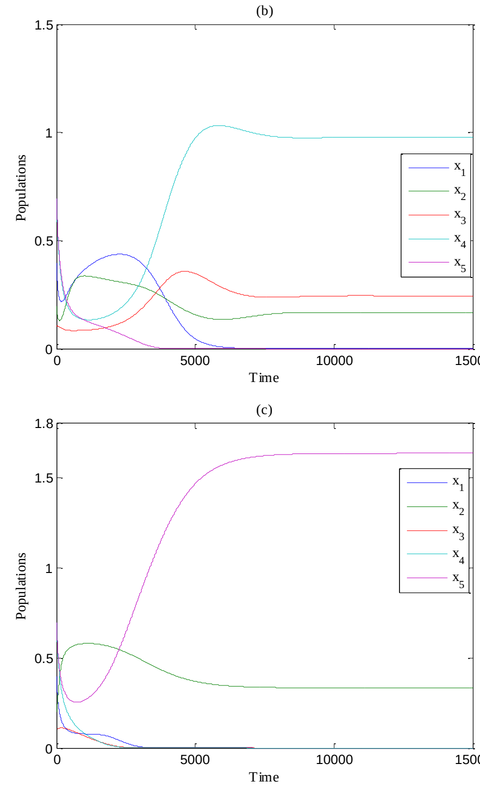


***Fig. 3:** Time series of the solution of system (2) for the data given by Eq. (30) with different values of* u_1 *. (a) System (2) approaches to* $E_6 = (0.1, 0, 0.2, 0.8, 0)$ *for* $u_1 = 1.1$ *. (b) System (2) approaches to* $E_7 = (0, 0.1, 0.2, 0.8, 0)$ *for* $u_1 = 1.5$ *. (c) System (2) approaches to* $E_5 = (0, 0.3, 0, 0, 0.7)$ *for* $u_1 = 1.8$ *.*

On the other hand varying the parameter u_2 keeping the rest of parameters values as in Eq. (30), it observed that for $0.79 \leq u_2 \leq 1.04$, the solution of system (2) approaches asymptotically to $E_7 = (0, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, 0)$, while for $u_2 < 0.79$ the solution approaches asymptotically to $E_5 = (0, \breve{x}_2, 0, 0, \breve{x}_5)$. Moreover for the data given by Eq. (30) with $u_3 \geq 1.51$, the solution of system (2) approaches asymptotically to $E_6 = (\bar{\bar{x}}_1, 0, \bar{\bar{x}}_3, \bar{\bar{x}}_4, 0)$. In addition, varying the parameter u_4 in the range $u_4 \geq 1.23$ with other data as in Eq.(30) the solution of system (2) approaches asymptotically to $E_6 = (\bar{\bar{x}}_1, 0, \bar{\bar{x}}_3, \bar{\bar{x}}_4, 0)$ too, however for $u_4 \leq 1.05$, it is observed that the solution of system (2) approaches asymptotically to the equilibrium point $E_7 = (0, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, 0)$. All these cases can be represented in figures similar to those shown in Fig. (3), with slightly difference in the position of equilibrium points. Similarly, for the data given by Eq. (30) with one of the following ranges at a time $u_8 \geq 1.12$; $u_9 \geq 0.54$; $u_{11} \leq 0.11$; $u_{13} \geq 1.16$; $u_{14} \geq 0.1$; $u_{15} \geq 0.22$; $u_{16} \geq 0.6$; $u_{17} \leq 0.7$ or $e_5 \geq 0.7$ it is observed that the trajectory of system (2) approaches asymptotically to the third three species equilibrium point $E_8 = (\tilde{\tilde{x}}_1, \tilde{\tilde{x}}_2, 0, 0, \tilde{\tilde{x}}_5)$ as explained in the typical figure represented by Fig. (2) with slightly difference in the

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position of point. Now, for the parameters values given in Eq. (30) with varying the following three parameters simultaneously $u_8 \geq 0.45$, $u_{14} \geq 0.2$ and $u_{18} \geq 0.2$, it is observed that the solution of system (2) approaches asymptotically to the first two species equilibrium point $E_3 = (\bar{x}_1, \bar{x}_2, 0,0,0)$ as shown in the typical figure given by Fig. (4).

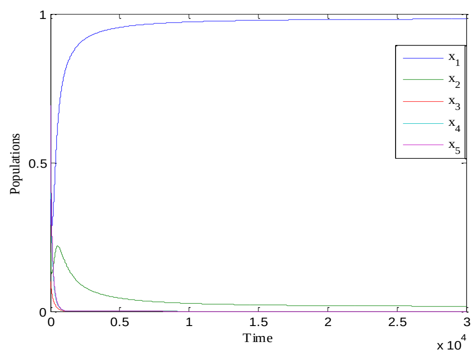

Fig. 4: *Time series of the solution of system (2), for the data given by Eq. (30) with* $u_8 = 0.5$, $u_{14} = 0.25$ *and* $u_{18} = 0.3$, *that approaches asymptotically to* $E_3 = (0.9, 0.01, 0, 0, 0)$.

However, for the parameters values given in Equation (30) with varying the following two parameters simultaneously $u_5 \geq 1.3$ and $u_{14} \geq 0.11$, it is observed that the solution of system (2) approaches asymptotically to the second two species equilibrium point $E_4 = (\hat{x}_1, 0,0,0, \hat{x}_5)$ as shown in typical figure given by Fig. (5).

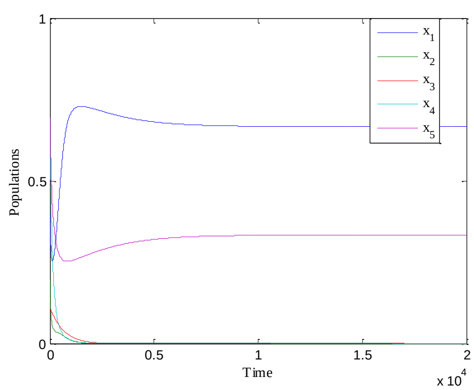

Fig. 5: *Time series of the solution of system (2), for the data given by Eq. (30) with* $u_5 = 1.5$ *and* $u_{14} = 0.15$, *that approaches asymptotically to* $E_4 = (0.6, 0, 0, 0, 0.3)$.

Now, for the parameters values given in Eq. (30) with varying the following three parameters simultaneously $u_2 \leq 0.9$, $u_{14} \geq 0.45$ and $u_{18} \geq 0.35$, it is observed that the solution of system (2) approaches asymptotically to the equilibrium point $E_2 = (0, \frac{1}{u_2}, 0,0,0)$ as shown in the typical figure given by Fig. (6).

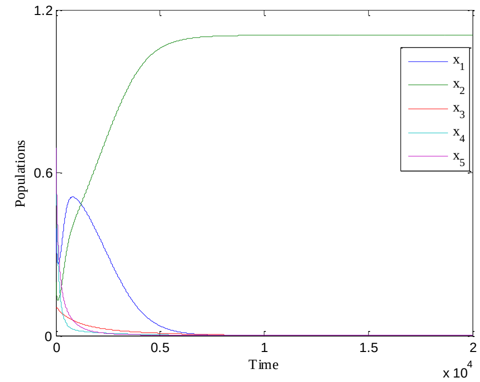

Fig. 6: *Time series of the solution of system (2), for the data given by Eq. (30) with* $u_2 = 0.8$, $u_{14} = 0.5$ *and* $u_{18} = 0.4$, *that approaches asymptotically to* $E_2 = (0, 1.11, 0, 0, 0)$.

Finally, for the parameters values given in Eq. (30) with varying the following three parameters simultaneously $u_3 \geq 1.25$, $u_{14} \geq 0.4$ and $u_{18} \geq 0.2$, it is observed that the solution of system (2) approaches asymptotically to the equilibrium point $E_1 = (1,0,0,0,0)$ as shown in the typical figure given by Fig. (7).

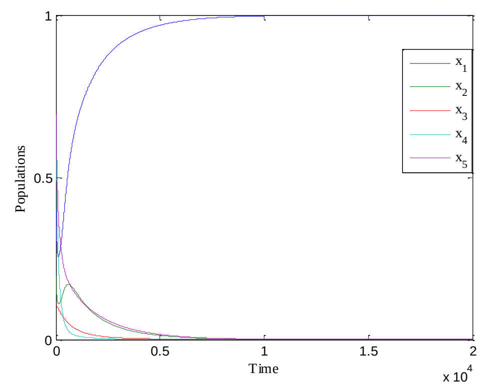

Fig. 7: *Time series of the solution of system (2), for the data given by Eq. (30) with* $u_3 = 1.3$, $u_{14} = 0.5$ *and* $u_{18} = 0.4$, *that approaches asymptotically to* $E_1 = (1,0,0,0,0)$

6. Conclusions and discussion

In this paper, we proposed and analyzed an ecological model that described the dynamical behavior of the food web real system. The model included five non-linear autonomous differential equations that describe the dynamics of five different populations, namely first prey (N_1), second prey (N_2), immature predator (N_3), mature predator (N_4) and N_5 which is represent the top predator. The boundedness of system (2) has been discussed. The existence conditions of all possible equilibrium points are obtained. The local as well as global stability analyses of these points are carried out. Finally, numerical simulation is used to specific the control set of parameters that affect the dynamics of the system and confirm our obtained analytical results. Therefore system (2) has been solved numerically for different sets of initial points and different sets of parameters starting with the hypothetical set of data given by Eq. (30), and the following observations are obtained.

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1) System (2) do not has periodic dynamic, instead of that the solution of system (2) approaches asymptotically to one of its equilibrium point.
2) Decreasing the growth rate of the second prey, u_1, under a specific value leads to destabilized E_9 and the solution approaches to E_6. However increasing the value of u_1 above a specific value leads the system to approaches to E_7, Further increasing this parameter makes the system approaches to E_5.
3) Decreasing the value of intra specific competition between the individuals of second prey, u_2, under a specific value leads the system to approaches to E_7, Further decreasing this parameter makes the system approaches to E_5.
4) Increasing the parameter that describe the intensity of competition of the first prey to the second prey, u_3, above a specific value leads to destabilizing of E_9 and the solution approaches to E_6.
5) Decreasing the value of attack rate of mature predator to the second prey species, u_4, under a specific value leads the system to approaches to E_7, However increasing this parameter above a specific value makes the system approaches to E_6.
6) Decreasing the value of growth rate of the mature predator due to its feeding on the first prey, u_{11}, under a specific value makes the solution of system (2) approaches asymptotically to E_8 . The system has similar behavior in case of decreasing u_{17}.
7) Increasing the value of gown up rate of the immature predator, u_8, above a specific value makes the solution of system (2) approaches asymptotically to E_8 . The system has similar behavior in case of increasing the value of u_9, u_{13}, u_{14}, u_{15}, u_{16} or e_5.
8) Finally, varying the parameters values $u_i, i = 5,6,7,10,12,18$, m and n don't have qualitative effect on the dynamical behavior of system (2) and the system still approaches to a top predator free equilibrium point E_9,

Keeping the above in view, all these outcomes depend on the hypothetical set of parameters values given by Eq. (30), different results may be obtained for different sets of data.

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