

Impending Role Of Period Three Points In Chaotic Functions Through Sturm Method And Matlab Programming

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ABSTRACT: *Chaos Theory and Dynamical Systems* has been considered as one of the most significant breakthroughs in Mathematics in this century. Its applications in a wide range of subjects including Physics, Biology, Chemistry, Ecology, Fluid Mechanics, Engineering, Economics etc., have made the field very attractive and important for the researchers from various disciplines. The salient feature of the chaos theory is that even simplest looking maps illustrate virtually every important phenomena that occur in the Dynamical Systems. The logistic map $f(x) = kx(1-x)$ is a popular example of such a behaviour. This family has been one of the simplest of the nonlinear maps, but it exhibits the various aspects of dynamical systems varying from stability to chaos. However this concept needs to be generalized. The quite unknown dynamics of a cubic family of functions defined by $f(x) = x^3 + \lambda x$ shows some interesting similarities as well as differences with this family of functions. The advent of modern computers has quickened the pace of development in the field of Chaos. Matlab programming can also play a vital role to visualize the inner properties of chaotic functions, including one of the three essential properties of chaotic functions viz. existence of period three points. This paper explains the existence of period three points for cubic family of functions through the Sturm method for finding the solution of the higher degree polynomials and Matlab programming.

Keywords : Chaotic Behaviour of a function, Bifurcation Diagram, Sturm Method, Period Three points, MATLAB programming

1 INTRODUCTION

A real quadratic family $[2]$ $f(x) = kx(1-x)$, $k \in \mathbb{R}$ (set of real numbers), has firmly indicated that even the simplest looking functions may have the complicated dynamics. This indication has put every function under suspicion and the simple looking cubic family functions of the form $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3 + \lambda x$, $\lambda \in \mathbb{R}$, are no exceptions. But it is very difficult to realize the dynamics of these functions as it involves some complicated equations to solve, which is not feasible by common tools. It does require a combination of theoretical proofs along with the help of MATLAB programming. Thus, MATLAB programming has been proving to be a boon in the field of Chaos Theory now.

2 CONVENTIONS AND NOTATIONS :

2.1 Following are some conventions and notations used in the future references:

- A function f is said to be of class C^r on I if $f^{(r)}(x)$, exists and is continuous at all $x \in I$.
- A function f is said to be smooth if it is of class C^1 .
- For intervals I and J in \mathbb{R} , A function $f: I \rightarrow J$ is said to be a homeomorphism if $f(x)$ is one – one , onto and continuous and $f^{-1}(x)$ is also continuous.
A function $f: I \rightarrow J$ is said to be a C^r -diffeomorphism if f is a C^r – homeomorphism such that f^{-1} is also C^r .
- Intermediate Value Theorem : If $f: [a, b] \rightarrow \mathbb{R}$ is continuous and $f(a) = x$, $f(b) = y$ then for any z between x and y , there exists c between a and b such that $f(c) = z$.
- Let $f: [a, b] \rightarrow [a, b]$ be a continuous function, then f has at least one fixed point in $[a, b]$.
- Limit Point : Let $S \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is a limit point of S if there is a sequence of points in S converging to x . S is said to be a closed set if it contains all its limit points.

- A set $S \subset \mathbb{R}$ is said to be an open set if for any $x \in S$, there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset S$.
- A set S along with all its limit points is said to be a closure of S , which is denoted by \overline{S} .
- A subset U of S is dense in S if $\overline{U} = S$.
- The forward orbit of a point x is the set of points $x, f(x), f^2(x), \dots$.
- A point x is a fixed point of $f(x)$ iff $f(x) = x$.
- A point x is a periodic point of a prime period n if n is the smallest positive integer such that $f^n(x) = x$.
- A fixed point p with $|f'(p)| > 1$ is called the repelling fixed point or a repeller or a source. A fixed point p with $|f'(p)| < 1$ is called the attracting fixed point or a sink. A fixed point p with $|f'(p)| = 1$ is called the non-hyperbolic fixed point.
- Topologically Transitive Maps : A function $f: J \rightarrow J$, $J \subset \mathbb{R}$, is said to be topologically transitive if for any pair of open sets $U, V \subset J$, there exist some $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.
- Sensitive Dependence on initial conditions : A function $f: J \rightarrow J$, $J \subset \mathbb{R}$, is said to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that for any $x \in J$ and any neighbourhood N of x , there exists some $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.
- Chaotic Function : Let $V \subset \mathbb{R}$. A function $f: V \rightarrow V$ is said to be chaotic on V if :
 - 1) f has sensitive dependence on initial conditions
 - 2) f is topologically transitive and
 - 3) periodic points are dense in V .

2.2 Chaotic behavior of cubic polynomials

- **Theorem:** For a real function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3 + \lambda x$, period doubling route to chaos exists. The bifurcation diagram of $f(x)$ also exists. Proof : For $f(x) = x^3 + \lambda x$, $-1.5 \leq \lambda$

< -1 , period two points satisfy the equation $f^2(x) = x$. Solutions of this equation exist and they are given by $\pm \sqrt{-1-\lambda}$. However, there are no more periodic points of f except these period two points and the fixed points of f for $-1.5 \leq \lambda < -1$. But as λ slides in the left of -1.5 and moves towards -3 , the condition changes rapidly. More and more periodic points start appearing and by the time we reach to $\lambda = -3$, we find periodic points of all order for f . In fact, we can see that periodic points become dense for $\lambda = -3$. After the availability of period two points for $f(x) = x^3 + \lambda x$, $-1.5 \leq \lambda < -1$, one comes across period four points first, as λ moves further towards -3 . Then as λ moves further and further to -3 , we see period eight points, period sixteen points.....and so on. This process is popularly known as a 'period doubling route to chaos' [1]. This period doubling route can be visualized in the graph called the 'bifurcation diagram'. A bifurcation diagram shows the values of higher iterations of f for some point x . The following is a bifurcation diagram for cubic function $f(x) = x^3 + \lambda x$ showing the higher iterations (say, 200^{th} , 201^{st} , 202^{nd} and so on) for a point, say x , with respect to the values of λ . Since bifurcation diagram involves a great amount of calculations, normal computer techniques are not efficient enough to draw this diagram of the graph clearly. So, MATLAB programs are used to draw this bifurcation diagram for the cubic family $f(x) = x^3 + \lambda x$, $-3 \leq \lambda < -1.5$. The following are the MATLAB programs which combinly give the bifurcation diagram for the cubic family $f(x) = x^3 + \lambda x$, $-3 \leq \lambda < -1.5$.

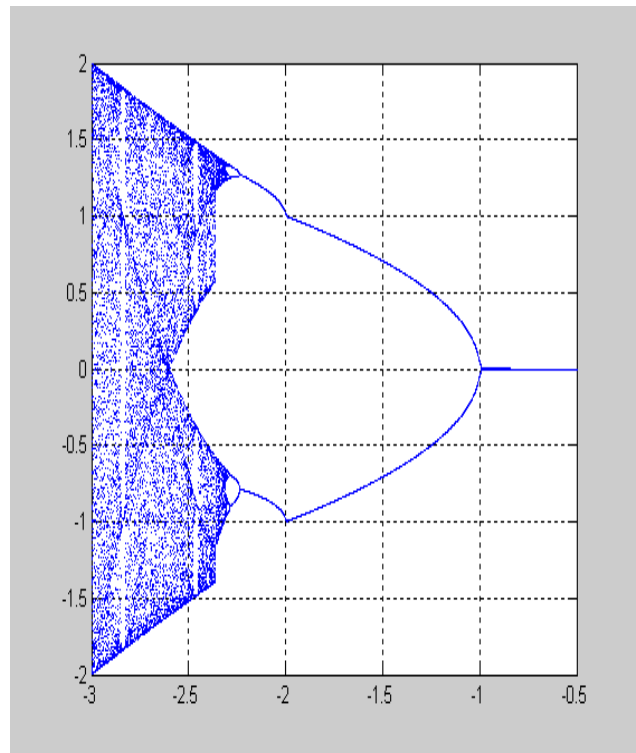
Program: 1:

```
function [y]=f(x,lem,n)
for k=1:n
    y=x^3+lem*x;
x=y;
end
```

Program: 2:

```
function bifurcation1(x);
for lem = -0.5:-0.001:-3
    [y] = f(x,lem,200);
    for i=1:20
        [z]=f(y,lem,1);
        plot(lem,z,'r');
        hold on;
        y=z;
    end
end
grid on;
```

The above two computer programs are used to plot the following bifurcation diagram for the cubic family $f(x) = x^3 + \lambda x$, $-3 \leq \lambda < -1.5$, starting with $x = 0.5$ and $\lambda = -0.5$. They actually give a geometrical plot for different values of λ (taken from -0.5 to -3 with the increment of -0.001) with respect to the iterations of a point after the 200^{th} iteration. A bifurcation diagram for a point $x = 0.5$ is as shown in the following figure. Here the horizontal axis represent the values of λ and the vertical axis represent the values of $f^{n+200}(0.5)$ ($f^{201}(0.5)$, $f^{202}(0.5)$, $f^{203}(0.5)$and so on).



➤ (A bifurcation diagram showing that as λ goes towards -3 , higher iterations of $x = 0.5$ change pattern. First they are mapped to 0 , then oscillate between two values, then oscillate between four values and so on.)

Looking from the right side, a straight line in the beginning shows that for the values of λ between -0.5 and -1 , the later iterations $f^{n+200}(0.5)$ (after 200^{th} iteration) of the point $x = 0.5$ have a constant value 0 . For the next range of λ (to the left of -1), the later iterations (after 200^{th} iteration) of the point $x = 0.5$ oscillate between two values. At about $\lambda \approx -2.232$, we see a change in the dynamics. Now we see that the iterations of $x = 0.5$ start oscillating by taking four values. This process of period doubling seems to continue endlessly as λ moves towards -3 . It is observed that for the cubic family $x^3 + \lambda x$, $\lambda \in \mathbb{R}$, period two points are born at $\lambda = -1$, period four points are born at about $\lambda \approx -2.232$, period eight points are born at about $\lambda \approx -2.289$. Let $a_1 =$ period two bifurcation value of $\lambda = -1$, $a_2 =$ period four bifurcation value of $\lambda \approx -2.232$, $a_3 =$ period eight bifurcation value of $\lambda \approx -2.289$. Then $(a_1 - a_2) / (a_2 - a_3) \approx 21.6140351$. Similarly we can get the ratios $(a_{k-1} - a_k) / (a_k - a_{k+1})$ for higher values of k . In the case of quadratic family $k x (1-x)$, the limiting value of $(a_{k-1} - a_k) / (a_k - a_{k+1})$ as $k \rightarrow \infty$ exists and found to be a fixed number, denoted by δ , whose approximate value is 4.6692016091 . This number δ is known as a Feigenbaum Constant. While studying the period doubling process and bifurcations for the quadratic functions $k x (1-x)$, $k \in \mathbb{R}$, Michel Feigenbaum found this number for the period doubling bifurcation.

Period 3 points for the cubic family through MATLAB programming :

Existence of period 3 points for the cubic family $f(x) = x^3 + \lambda x$ gives an important clue about the chaotic behaviour of the family. But to find out such points with usual mathematical methods is very close to impossible. A graphical view suggests

that as the decreasing value of λ crosses the value (almost equal to) -2.6996, the period three points for the cubic family exist. However, to justify this fact theoretically, it requires to use some less known methods about polynomials and an effective use of Matlab programming. In fact, the period 3 points for the cubic family are contained in the set of fixed points of the polynomial equation $f^{(3)}(x) = x$. This polynomial equation is of degree 27 as given by : $x^{27} + (9\lambda)x^{25} + (36\lambda^2)x^{23} + (84\lambda^3 + 3\lambda)x^{21} + (126\lambda^4 + 21\lambda^2)x^{19} + (126\lambda^5 + 63\lambda^3)x^{17} + (84\lambda^6 + 105\lambda^4 + 3\lambda^2)x^{15} + (36\lambda^7 + 105\lambda^5 + 15\lambda^3)x^{13} + (9\lambda^8 + 63\lambda^6 + 30\lambda^4)x^{11} + (\lambda^9 + 21\lambda^7 + 30\lambda^5 + \lambda^3 + \lambda)x^9 + (3\lambda^8 + 15\lambda^6 + 3\lambda^4 + 3\lambda^2)x^7 + (3\lambda^7 + 3\lambda^5 + 3\lambda^3)x^5 + (\lambda^6 + \lambda^4 + \lambda^2)x^3 + \lambda^3x = x$. Since every fixed point of $f(x)$ also satisfies the equation $f^{(3)}x = x$, the fixed points $x = 0$, $x = P_\lambda (= \sqrt{1-\lambda})$ and $x = -P_\lambda$ are the trivial solutions of the above equation of degree 27. (In fact, these are the only solutions of the equation $f^{(3)}x = x$, which are not period three points of the family.) Hence, x , $x - P_\lambda$ and $x + P_\lambda$ are factors of the 27-degree polynomial on the left hand side of the above equation. Dividing the above equation of degree 27 by $x(x - P_\lambda)(x + P_\lambda)$, the following polynomial equation of degree 24 is obtained :

$$x^{24} + a_1 x^{22} + a_2 x^{20} + a_3 x^{18} + a_4 x^{16} + a_5 x^{14} + a_6 x^{12} + a_7 x^{10} + a_8 x^8 + a_9 x^6 + a_{10} x^4 + a_{11} x^2 + a_{12} = 0 \tag{A}$$

Where $a_1 = 9\lambda + p^2$, $a_2 = 36\lambda^2 + p^2(a_1)$, $a_3 = 84\lambda^3 + 3\lambda + p^2(a_2)$

$a_4 = 126\lambda^4 + 21\lambda^2 + p^2(a_3)$, $a_5 = 126\lambda^5 + 63\lambda^3 + p^2(a_4)$

$a_6 = 84\lambda^6 + 105\lambda^4 + 3\lambda^2 + p^2(a_5)$, $a_7 = 36\lambda^7 + 105\lambda^5 + 15\lambda^3 + p^2(a_6)$

$a_8 = 9\lambda^8 + 63\lambda^6 + 30\lambda^4 + p^2(a_7)$, $a_9 = \lambda^9 + 21\lambda^7 + 30\lambda^5 + \lambda^3 + \lambda + p^2(a_8)$

$a_{10} = 3\lambda^8 + 15\lambda^6 + 3\lambda^4 + 3\lambda^2 + p^2(a_9)$, $a_{11} = 3\lambda^7 + 3\lambda^5 + 3\lambda^3 + p^2(a_{10})$

and $a_{12} = \lambda^6 + \lambda^4 + \lambda^2 + p^2(a_{11})$.

Now it is obvious that every solution of this polynomial equation of degree 24 is a period three point of the cubic family. For the polynomials of degree higher than four, it is usually hopeless to think of solving them exactly. But there are some efficient methods to find the close solutions of such a high degree polynomial ; like a Fourier-Budan method, Sturm method etc. In this section, the Sturm method is discussed which is a difficult but appropriate method. This method is well described in the book "Polynomials" by E. J. Barbeau. Now we discuss the Sturm method to justify the existence of period three points for the cubic family :

Sturm Method for the solution of the higher degree polynomials:

o **Method :** Let $p(x)$ be a given polynomial with real coefficients. Now define a sequence

{ $p_0(x)$, $p_1(x)$, $p_2(x)$, $p_3(x)$, } as follows :

$p_0(x) = p(x)$

$p_1(x) = p'(x) = \text{derivative of } p(x)$

$p_2(x) = p_1(x) q_1(x) - p_0(x) = \text{the negative of the remainder when } p_0(x) \text{ is divided by } p_1(x)$

$p_3(x) = p_2(x) q_2(x) - p_1(x) = \text{the negative of the remainder when } p_1(x) \text{ is divided by } p_2(x)$

$p_4(x) = p_3(x) q_3(x) - p_2(x) = \text{the negative of the remainder when } p_2(x) \text{ is divided by } p_3(x)$

Thus, at each stage for $k \geq 2$, $p_k(x)$ is the negative of the remainder when $p_{k-2}(x)$ is divided by $p_{k-1}(x)$. If there are multiple roots, the last nonzero remainder will not be constant. This sequence of polynomials is said to be a Sturm sequence for $p(x)$. Now suppose there are two numbers u and v such that the aim is to know the number of real roots of $p(x)$ in the interval (u, v) . Let U be the number of sign changes in the sequence { $p_0(u)$, $p_1(u)$, $p_2(u)$, $p_3(u)$, } and V be the number of sign changes in the sequence { $p_0(v)$, $p_1(v)$, $p_2(v)$, $p_3(v)$, }. Then Sturm theorem says that the number of real roots of $p(x)$ between u and v (with each multiple root counted only once) is exactly $U - V$. → Now we switch over to the main problem of finding the roots of the polynomial of degree 24 to get the period three points of the cubic family. For a polynomial of degree 24, applying Sturm's method manually is a very troublesome job. Hence, a help of Matlab is taken. The following is a Matlab programme for the Sturm method to find the number of real solutions of a polynomial of degree 24 (i.e. the number of period three points for the cubic family). This programme calculates all the values for the Sturm sequence and finds the value of $U - V$ at the end. This programme also finds all the solutions of the polynomial of degree 24 (i.e., actual period three points of a cubic family) for the different values of lambda.

Matlab Programme for the Sturm Method

```
function sturm(k);
format long;
p=sqrt(1-k);

u=-p;
v=p;

a1=9*k+p^2;
a2=36*k^2+(p^2)*a1;a3=84*k^3+3*k+(p^2)*a2;a4=126*k^4+21*k^2+(p^2)*a3;a5=126*k^5+63*k^3+(p^2)*a4;a6=84*k^6+105*k^4+3*k^2+(p^2)*a5;a7=36*k^7+105*k^5+15*k^3+(p^2)*a6;a8=9*k^8+63*k^6+30*k^4+(p^2)*a7;a9=k^9+21*k^7+30*k^5+k^3+k+(p^2)*a8;a10=3*k^8+15*k^6+3*k^4+3*k^2+(p^2)*a9;a11=3*k^7+3*k^5+3*k^3+(p^2)*a10;a12=k^6+k^4+k^2+(p^2)*a11;
p0=[1,0,a1,0,a2,0,a3,0,a4,0,a5,0,a6,0,a7,0,a8,0,a9,0,a10,0,a11,0,a12];

X=p;

p1=[24,0,22*a1,0,20*a2,0,18*a3,0,16*a4,0,14*a5,0,12*a6,0,10*a7,0,8*a8,0,6*a9,0,4*a10,0,2*a11,0];

c=1;
```

```
a(c)=polyval(p0,u);  
b(c)=polyval(p0,v);  
  
for i=1:24  
    [q,r]=deconv(p0,p1);  
    p0=p1;  
    p1=-r(3:length(r));  
  
    c=c+1;  
  
    a(c)=polyval(p0,u);  
    b(c)=polyval(p0,v);  
end
```

```
U=0;  
V=0;
```

```
for i=1:length(a)-1  
  
    if (a(i)>0 & a(i+1)<0) | (a(i)<0 & a(i+1)>0)  
        U=U+1;  
    end  
  
    if (b(i)>0 & b(i+1)<0) | (b(i)<0 & b(i+1)>0)  
        V=V+1;  
    end  
  
end
```

```
disp('Number of real roots of the polynomial is: ')  
disp(U-V)  
disp('And these real roots are as follows:');  
disp(roots(X))
```

The above programs confirms the existence of period three points of cubic functions and leads to prove that cubic polynomials $f(x) = x^3 + \lambda x$, $\lambda \leq -3$ is chaotic. Thus, MATLAB programming plays a crucial role to prove that some simple looking real functions are, in fact, chaotic functions.

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