ISSN 2347-4289

The Effect Of External Source Of Disease On The HIV\ Aids Model With Bifurcation

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ABSTRACT: In this paper a mathematical model that describes the spread of sexual infectious disease in a population is proposed and studied. It is assumed that the disease divided the population into four classes: susceptible individuals of males (*S*), infected individuals of males (*I*), susceptible individuals of females (S^*) and infected individuals of females (I^*). The impact of contact between of population and external sources of disease for example (blood and other), on the dynamics of SIS^*I^* epidemic model is investigated. The existence, uniqueness and boundedness of the solution of the model are discussed. The local and global stability of the model is studied. The occurrence of local bifurcation in the model is investigated. Finally the global dynamics of the proposed model is studied numerically.

Keywords: Epidemic models, Stability, HIV\AIDS, external sources, Local bifurcation.

1. INTRODUCTION

Sexually transmitted diseases are one of the serious diseases faced on human life at the moment due to the spread of disease, the difficulty to control it, and the lack of its treatment. One of these diseases is the disease of HIV / AIDS which causes the death and Killing about 2 million people all over the world. While, the number of new infections may be even more than 2 million, which shows that the problem in the spread of the disease more severe and dangerous in the future. Moreover, the vast majority of the populations living To HIV / AIDS are female, where women represent 50% of people in most developed countries [1]. The first mathematical model that describes sexual diseases, or diseases transmitted through sex is Cook and Yorke [2] and systems for the disease was one-sex model. Lajmanovich and Yorke [3], Studied a special system for gonorrhea consists of two-sex model. To address the seriousness of the disease and provide a better understanding and more clarity predictions of the behavior of the spread of the disease has been the development of many forms of mathematical models and applied to the epidemic of HIV / AIDS, for example, Knox, E. G. [4] is studied the transmission of AIDS. Anderson, R. M., Medley, G.; May, F.R. M. and Johnson, A. M. A. [5] show a preliminary study of the dynamics of AIDS. Anderson, R. M. [6], the role of mathematical models in the study of the transmission of AIDS. Dietz, K.; Heesterbeek, J. A. P. and Tudor,D.W. [7], the effects of infection with HIV. Dietz, K. [8] studied Transmission Dynamics of HIV. Brauer, F. and Castillo-Chavez, C. [9] studied a group of Mathematical Models of AIDS disease. And Levin, B.R., Bull, J.J. and Stewart, F.M. [10] offered study of Evolution, and Future of the HIV/AIDS Pandemic. In this paper we proposed and studied a mathematical model consisting of SIS^*I^* epidemic model, it is assumed that the disease transmitted by contact as well as external sources. The local as well as global stability analysis of the model is investigated. Also, the local bifurcation is discussed.

2. The mathematical model

Consider a simple epidemiological model in which the total population (say *N(t)*) at time t is divided in to fore sub classes the susceptible individuals of males *S(t),* susceptible individuals of females $S^*(t)$, infected individuals of males *I(t)* and infected individuals of females $I^*(t)$ which represented in the block diagram given by figure (1).

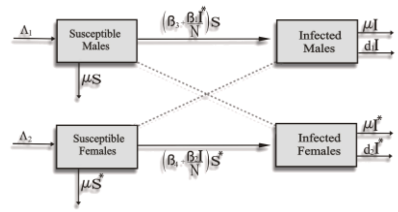

Figure (1): *Block diagram of system (1).*

Can be represented by the following system of nonlinear ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda_1 - \left(\beta_3 + \frac{\beta_1 I^*}{N}\right)S - \mu S \\
\frac{dI}{dt} &= \left(\beta_3 + \frac{\beta_1 I^*}{N}\right)S - (\mu + d_1)I \\
\frac{dS^*}{dt} &= \Lambda_2 - \left(\beta_4 + \frac{\beta_2 I}{N}\right)S^* - \mu S^* \\
\frac{dI^*}{dt} &= \left(\beta_4 + \frac{\beta_2 I}{N}\right)S^* - (\mu + d_2)I^*
\end{aligned} \qquad (1)$$

Here $\Lambda_i > 0, i = 1,2$ are recruitment rate of the population (Males and Females) respectively, $\mu > 0$ is the natural death rate of each population, $d_i > 0, i = 1,2$ are the disease related death rate, $\beta_i > 0, i = 1,2$ are the infected rate (incidence rate) of the susceptible individuals (Males and Females) respectively due- to directed contact with the infected individuals and we assumed that the disease in the above model will transmitted between the population individuals by contact as well as external sources of disease for example (Blood, Medical tools, etc) with an external incidence rate $\beta_i > 0, i = 3,4$. Therefore at any point of time t the total number of population becomes $N = S(t) + I(t) + S^*(t) + I^*(t)$. Obviously, due to the biological meaning of the variables $S(t), I(t), S^*(t)$ and $I^*(t)$, in system (1) has the domain

$R_+^4 = \{(S, I, S^*, I^*) \in R_+^4, S \geq 0, I \geq 0, S^* \geq 0, I^* \geq 0\}$. This is positively invariant for system (1). Clearly, the interaction functions on the right hand side of system (1) are continuously differentiable. In fact they are Liptschizan function on R_+^4. Therefore the solution of system (1) exists and unique. Further, all solutions of the system (1) with non-negative initial conditions are uniformly bounded as shown in the following theorem.

Theorem (1) : All the solutions of system (1), which are initiate in R_+^4, are uniformly bounded.

Proof: Let $\left(S(t), I(t), S^*(t), I^*(t)\right)$ be any solution of the system (1) with non-negative initial conditions $\left(S(0), I(0), S^*(0), I^*(0)\right)$. Since $N = S(t) + I(t) + S^*(t) + I^*(t)$, then:

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dS^*}{dt} + \frac{dI^*}{dt}$$

This gives

$$\frac{dN}{dt} = \Lambda_1 + \Lambda_2 - \mu\left(S + I + S^* + I^*\right) - d_1 I - d_2 I^*$$

So, $\frac{dN}{dt} + \mu N \leq \Lambda_1 + \Lambda_2$

Now, by using Gronwall Lemma [11], it obtains that:

$$N(t) \leq \frac{\Lambda_1 + \Lambda_2}{\mu}(1 - e^{-\mu t}) + N(0)e^{-\mu t}$$

Therefore, $N(t) \leq \frac{\Lambda_1 + \Lambda_2}{\mu}$, as $t \to \infty$, hence all the solutions of system (1) that initiate in R_+^4 are confined in the reign:

$$\Gamma = \left\{(S, I, S^*, I^*) \in R_+^4 : N \leq \frac{\Lambda_1 + \Lambda_2}{\mu}\right\}$$

Which complete the proof. ■

3. Existence of equilibrium point of system (1)

The system (1) has at most fore biologically feasible points, namely $E_i = (S_i, I_i, S_i^*, I_i^*), i = 0,1,2,3$. The existence conditions for each of these equilibrium points are discussed in the following:

1) If $I = I^* = 0$, then the system (1) has an equilibrium point called a disease free equilibrium point and denoted by $E_0 = (S_0, 0, S_0^*, 0)$ where:

$$\left.\begin{aligned} S_0 &= \frac{\Lambda_1}{\mu} \\ S_0^* &= \frac{\Lambda_2}{\mu} \end{aligned}\right\} \quad (2)$$

2) If $I = 0$, then the system (1) has equilibrium point called a male's disease free equilibrium point and denoted by $E_1 = (S_1, 0, S_1^*, I_1^*)$ where S_1, S_1^* and I_1^* represented the positive solution of following set of equations:

$$\begin{aligned} &\Lambda_1 - \mu S = 0 \\ &\Lambda_2 - (\beta_4 + \mu)S^* = 0 \\ &\beta_4 S^* - (\mu + d_2)I^* = 0 \end{aligned} \quad (3)$$

From equation (1) and (2) of above system we get:

$$\left.\begin{aligned} S_1 &= \frac{\Lambda_1}{\mu} \\ S_1^* &= \frac{\Lambda_2}{\beta_4 + d_2} \end{aligned}\right\} \quad (4a)$$

Now, substituting S_1^* in equation (3) of system (3) we get:

$$I_1^* = \frac{\beta_4 \Lambda_2}{(\beta_4 + \mu)(d_2 + \mu)} \quad (4b)$$

3) If $I^* = 0$, then the system (1) has equilibrium point called a female's disease free equilibrium point and denoted by $E_2 = (S_2, I_2, S_2^*, 0)$ where S_2, I_2, S_2^* represented the positive solution of the following set of equations:

$$\begin{aligned} &\Lambda_1 - (\beta_3 + \mu)S = 0 \\ &\beta_3 S - (\mu + d_1)I = 0 \\ &\Lambda_2 - \mu S^* = 0 \end{aligned} \quad (5)$$

From equations (1) and (3) of above system we get:

$$\left.\begin{aligned} S_2 &= \frac{\Lambda_1}{\beta_3 + \mu} \\ S_2^* &= \frac{\Lambda_2}{\mu} \end{aligned}\right\} \quad (6a)$$

Now, substituting S_2 in equation (2) of system (5) we get:

$$I_2 = \frac{\Lambda_1 \beta_3}{(\beta_3+\mu)(d_1+\mu)} \qquad (6b)$$

4) If $I \neq 0$ and $I^* \neq 0$ then the system (1) has an equilibrium point called endemic equilibrium point and denoted by $E_3 = (S_3, I_3, S_3^*, I_3^*)$ where S_3, I_3, S_3^* and I_3^* represented the positive solution of the following set of equations:

$$\begin{aligned}
&\Lambda_1 - \left(\beta_3 + \frac{\beta_1 I^*}{N}\right)S - \mu S = 0 \\
&\left(\beta_3 + \frac{\beta_1 I^*}{N}\right)S - (\mu + d_1) I_3 = 0 \\
&\Lambda_2 - \left(\beta_4 + \frac{\beta_2 I}{N}\right)S^* - \mu S^* = 0 \\
&\left(\beta_4 + \frac{\beta_2 I}{N}\right)S^* - (\mu + d_2) I_3^* = 0
\end{aligned} \qquad (7)$$

Straightforward computation to solve the above system of equations and from equation (1), (2) and (3) of system (7) gives that:

$$\left.\begin{aligned}
S_3 &= \frac{\Lambda_1 N}{\beta_3 N + \beta_1 I_3^* + \mu N} \\
I_3 &= \frac{\Lambda_1(\beta_3 N + \beta_1 I_3^*)}{(\mu + d_1)(\beta_3 N + \beta_1 I_3^* + \mu N)} \\
S_3^* &= \frac{\Lambda_2 N}{\beta_4 N + \beta_2 I + \mu N}
\end{aligned}\right\} \qquad (8)$$

Now, substituting I_3 and S_3^* in equation (4) of system (7) we get:

$$I_3^* = \frac{-D_2}{2D_1} + \frac{1}{2D_1}\sqrt{D_2^2 - 4D_1D_3} \qquad (9)$$

Here

$$\begin{aligned}
D_1 &= \beta_1(\mu + d_2)\left[\Lambda_1\beta_2 + N(\beta_4+\mu)(d_1+\mu)\right] > 0 \\
D_2 &= N(\mu + d_1)\left[\Lambda_1\beta_2\beta_3 + N(\beta_4+\mu)(\mu+d_1)(\beta_3+\mu)\right] \\
&\qquad - \Lambda_2\beta_1\beta_2\left[\Lambda_1 + N(\mu+d_1)\right] \\
D_3 &= \Lambda_2\beta_2 N\left[\Lambda_1\beta_3 + N(\mu+d_1)(\beta_3+\mu)\right] < 0
\end{aligned}$$

Clearly, equation (9) has a unique positive root by I_3^* and then (E_3) exists uniquely in Int. R_+^4 if and only if $D_2 > 0$.

4. Local stability analysis of system (1)

In this section, the local stability analysis of the equilibrium points $E_i, i = 0,1,2,3$ of the system (1) studied as shown in the following theorems.

Theorem (2): The disease free equilibrium point $E_0 = (S_0, 0, S_0^*, 0)$ of system (1) is locally asymptotically stable if the following condition is satisfied:

$$\mu N > Max.\left\{ 2\beta_2 S_0^* - d_1 N, 2\beta_1 S_0 - d_2 N \right\} \qquad (10)$$

Proof: The Jacobian matrix of system (1) at (E_0) can be written as:

$$J(E_0) = \begin{bmatrix}
-(\beta_3+\mu) & 0 & 0 & -\frac{\beta_1 S_0}{N} \\
\beta_3 & -(\mu+d_1) & 0 & \frac{\beta_1 S_0}{N} \\
0 & -\frac{\beta_2 S_0^*}{N} & -(\beta_4+\mu) & 0 \\
0 & \frac{\beta_2 S_0^*}{N} & \beta_4 & -(\mu+d_2)
\end{bmatrix}$$

Now, according to Gersgorin theorem if the following condition holds:

$$|a_{ii}| > \sum_{\substack{i=1 \\ i\neq j}}^{4} |a_{ij}|$$

Then all eigenvalues of $J(E_0)$ exists in the region:

$$\wp = \bigcup \left\{ U^* \in C : \left|U^* - a_{ii}\right| < \sum_{\substack{i=1 \\ i\neq j}}^{4} |a_{ij}| \right\}$$

Therefore, according to the given condition (10) all the eigenvalues of $J(E_0)$ exists in the left half plane and hence, E_0 is locally asymptotically stable. ■

Theorem (3): The male's disease free equilibrium point $E_1 = (S_1, 0, S_1^*, I_1^*)$ of system (1) is locally asymptotically stable if the following condition is satisfied:

$$\mu N > Max.\left\{ 2\beta_2 S_1^* - d_1 N, 2\beta_1 S_1 - d_2 N \right\} \qquad (11)$$

Proof: The Jacobian matrix of system (1) at (E_1) can be written as:

$$J(E_1) = \begin{bmatrix}
-(\beta_3 + \frac{\beta_1 I_1^*}{N} + \mu) & 0 & 0 & -\frac{\beta_1 S_1}{N} \\
\beta_3 + \frac{\beta_1 I_1^*}{N} & -(\mu+d_1) & 0 & \frac{\beta_1 S_1}{N} \\
0 & -\frac{\beta_2 S_1^*}{N} & -(\beta_4+\mu) & 0 \\
0 & \frac{\beta_2 S_1^*}{N} & \beta_4 & -(\mu+d_2)
\end{bmatrix}$$

ISSN 2347-4289

Now, according to Gersgorin theorem if the following condition holds:

$$|b_{ii}| > \sum_{\substack{i=1 \\ i\neq j}}^{4} |b_{ij}|$$

Then all eigenvalues of $J(E_1)$ exists in the region:

$$\varsigma = \bigcup \left\{ U^* \in C : \left| U^* - b_{ii} \right| < \sum_{\substack{i=1 \\ i\neq j}}^{4} |b_{ij}| \right\}$$

Therefore, according to the given condition (11) all the eigenvalues of $J(E_1)$ exists in the left half plane and hence, E_1 is locally asymptotically stable. ■

Theorem (4): The female's disease free equilibrium point $E_2 = (S_2, I_2, S_2^*, 0)$ of system (1) is locally asymptotically stable if the following condition is satisfied:

$$\mu N > Max.\left\{ 2\beta_2 S_2^* - d_1 N, 2\beta_1 S_2 - d_2 N \right\} \quad (12)$$

Proof : The Jacobian matrix of system (1) at (E_2) can be written as:

$$J(E_2) = \begin{bmatrix} -(\beta_3 + \mu) & 0 & 0 & -\frac{\beta_1 S_2}{N} \\ \beta_3 & -(\mu + d_1) & 0 & \frac{\beta_1 S_2}{N} \\ 0 & -\frac{\beta_2 S_2^*}{N} & -(\beta_4 + \frac{\beta_2 I_2}{N} + \mu) & 0 \\ 0 & \frac{\beta_2 S_2^*}{N} & \beta_4 + \frac{\beta_2 I_2}{N} & -(\mu + d_2) \end{bmatrix}$$

Now, according to Gersgorin theorem if the following condition holds:

$$|c_{ii}| > \sum_{\substack{i=1 \\ i\neq j}}^{4} |c_{ij}|$$

Then all eigenvalues of $J(E_2)$ exists in the region:

$$\omega = \bigcup \left\{ U^* \in C : \left| U^* - c_{ii} \right| < \sum_{\substack{i=1 \\ i\neq j}}^{4} |c_{ij}| \right\}$$

Therefore, according to the given condition (12) all the eigenvalues of $J(E_2)$ exists in the left half plane and hence, E_2 is locally asymptotically stable. ■

Theorem (5): The endemic equilibrium point $E_3 = (S_3, I_3, S_3^*, I_3^*)$ of system (1) is locally asymptotically stable if the following condition is satisfied:

$$\mu N > Max.\left\{ 2\beta_2 S_3^* - d_1 N, 2\beta_1 S_3 - d_2 N \right\} \quad (13)$$

Proof: The Jacobian matrix of system (1) at (E_3) can be written as:

$$J(E_3) = \begin{bmatrix} -(\beta_3 + \frac{\beta_1 I_3^*}{N} + \mu) & 0 & 0 & -\frac{\beta_1 S_3}{N} \\ \beta_3 + \frac{\beta_1 I_3^*}{N} & -(\mu + d_1) & 0 & \frac{\beta_1 S_3}{N} \\ 0 & -\frac{\beta_2 S_3^*}{N} & -(\beta_4 + \frac{\beta_2 I_3}{N} + \mu) & 0 \\ 0 & \frac{\beta_2 S_3^*}{N} & \beta_4 + \frac{\beta_2 I_3}{N} & -(\mu + d_2) \end{bmatrix}$$

Now, according to Gersgorin theorem if the following condition holds:

$$|d_{ii}| > \sum_{\substack{i=1 \\ i\neq j}}^{4} |d_{ij}|$$

Then all eigenvalues of $J(E_3)$ exists in the region:

$$\tau = \bigcup \left\{ U^* \in C : \left| U^* - d_{ii} \right| < \sum_{\substack{i=1 \\ i\neq j}}^{4} |d_{ij}| \right\}$$

Therefore, according to the given condition (13) all the eigenvalues of $J(E_3)$ exists in the left half plane and hence, E_3 is locally asymptotically stable. ■

5. Globally stability of system (1)

In this section, the global dynamics of system (1) is studied with the help of Lyapunov function as shown in the following theorems.

Theorem (6): Assume that, the disease free equilibrium point E_0 of system (1) is locally asymptotically stable. Then the basin of attraction of E_0, say $B(E_0) \subset R_+^4$, it is globally asymptotically stable if satisfy the following condition:

$$N\left[(\mu + d_1)I + (\mu + d_2)I^*\right] > \left[(\beta_3 N + \beta_1 I^*)S_0 + (\beta_4 N + \beta_2 I)S_0^*\right] \quad (14)$$

Proof: Consider the following positive definite function:

$$V_1 = \left(S - S_0 - S_0 \ln \frac{S}{S_0} \right) + I + \left(S^* - S_0^* - S_0^* \ln \frac{S*}{S_0^*} \right) + I^*$$

ISSN 2347-4289

Clearly, $V_1 : R_+^4 \to R$ is a continuously differentiable function such that $V_1(S_0, 0, S_0^*, 0) = 0,$ and $V_1(S, I, S^*, I^*) > 0,\ \forall (S, I, S^*, I^*) \neq (S_0, 0, S_0^*, 0)$. Further we have:

$$\frac{dV_1}{dt} = \left(\frac{S - S_0}{S}\right)\frac{dS}{dt} + \frac{dI}{dt} + \left(\frac{S^* - S_0^*}{S^*}\right)\frac{dS^*}{dt} + \frac{dI^*}{dt}$$

By simplifying this equation we get:

$$\frac{dV_1}{dt} = -\frac{\mu}{S}(S - S_0)^2 - \frac{\mu}{S^*}\left(S^* - S_0^*\right)^2 + S_0\left(\beta_3 + \frac{\beta_1 I^*}{N}\right) + S_0^*\left(\beta_4 + \frac{\beta_2 I}{N}\right) - (\mu + d_1)I - (\mu + d_2)I^*$$

Obviously, $\frac{dV_1}{dt} < 0$, and then V_1 is a Lyapunov function provided that condition (14) hold. Thus E_0 is globally asymptotically stable in the interior of $B(E_0)$, which means that $B(E_0)$ is the basin of attraction and that complete the proof. ■

Theorem (7): Assume that, the male's disease free equilibrium point E_1 of system (1) is locally asymptotically stable. Then the basin of attraction of E_1, say $B(E_1) \subset R_+^4$, it is globally asymptotically stable if satisfy the following conditions:

$$\left(\frac{\beta_4}{I^*}\right)^2 < 4\left(\frac{\beta_4 + \mu}{S^*}\right)\left(\frac{\mu + d_2}{I^*}\right) \quad \text{(15a)}$$

$$\left[(\beta_3 N + \beta_1 I^*)S_1 + \beta_2 S_1^* I\right] I^* < \left[(\mu + d_1)NI^* + \beta_2 I_1^* S^*\right] I \quad \text{(15b)}$$

Proof: Consider the following positive definite function:

$$V_2 = \left(S - S_1 - S_1 \ln\frac{S}{S_1}\right) + I + \left(S^* - S_1^* - S_1^* \ln\frac{S*}{S_1^*}\right) + \left(I^* - I_1^* - I_1^* \ln\frac{I^*}{I_1^*}\right)$$

Clearly, $V_2 : R_+^4 \to R$ is a continuously differentiable function such that $V_2(S_1, 0, S_1^*, I_1^*) = 0,$ and $V_2(S, I, S^*, I^*) > 0,\ \forall (S, I, S^*, I^*) \neq (S_1, 0, S_1^*, I_1^*)$. Further we have:

$$\frac{dV_2}{dt} = \left(\frac{S - S_1}{S}\right)\frac{dS}{dt} + \frac{dI}{dt} + \left(\frac{S^* - S_1^*}{S^*}\right)\frac{dS^*}{dt} + \left(\frac{I^* - I_1^*}{I^*}\right)\frac{dI^*}{dt}$$

By simplifying this equation we get:

$$\frac{dV_2}{dt} = -\frac{\mu}{S}(S - S_1)^2 - \left(\frac{\beta_4 + \mu}{S^*}\right)\left(S^* - S_1^*\right)^2 + \frac{\beta_4}{I^*}\left(S^* - S_1^*\right)\left(I^* - I_1^*\right) - \left(\frac{\mu + d_2}{I^*}\right)\left(I^* - I_1^*\right)^2 + \left(\beta_3 + \frac{\beta_1 I^*}{N}\right)S_1 - (\mu + d_1)I + \frac{\beta_2 I}{N}\left(S_1^* - \frac{S^* I_1^*}{I^*}\right)$$

Therefore, according to condition (15a) it is obtain that:

$$\frac{dV_2}{dt} \leq -\frac{\mu}{S}(S - S_1)^2 - \left[\sqrt{\frac{\beta_4 + \mu}{S^*}}\left(S^* - S_1^*\right) - \sqrt{\frac{\mu + d_2}{I^*}}\left(I^* - I_1^*\right)\right]^2 + \left(\beta_3 + \frac{\beta_1 I^*}{N}\right)S_1 - (\mu + d_1)I + \frac{\beta_2 I}{N}\left(S_1^* - \frac{S^* I_1^*}{I^*}\right)$$

Obviously, $\frac{dV_2}{dt} < 0$ for every initial points satisfying conditions (15b) and then V_2 is a Lyapunov function provided that conditions (15a)-(15b) hold. Thus E_1 is globally asymptotically stable in the interior of $B(E_1)$, which means that $B(E_1)$ is the basin of attraction and that complete the proof. ■

Theorem (8): Assume that, the female's disease free equilibrium point E_2 of system (1) is locally asymptotically stable. Then the basin of attraction of E_2, say $B(E_2) \subset R_+^4$, it is globally asymptotically stable if satisfy the following conditions:

$$\left(\frac{\beta_3}{I}\right)^2 < 4\left(\frac{\beta_3 + \mu}{S}\right)\left(\frac{\mu + d_1}{I}\right) \quad \text{(16a)}$$

$$\left[(\beta_4 N + \beta_2 I)S_2^* + \beta_1 S_2 I^*\right] I < \left[(\mu + d_2)IN + \beta_1 I_2 S\right] I^* \quad \text{(16b)}$$

Proof: Consider the following positive definite function:

$$V_3 = \left(S - S_2 - S_2 \ln\frac{S}{S_2}\right) + \left(I - I_2 - I_2 \ln\frac{I}{I_2}\right) + \left(S^* - S_2^* - S_2^* \ln\frac{S*}{S_2^*}\right) + I^*$$

C

learly, $V_3 : R_+^4 \to R$ is a continuously differentiable function such that $V_3(S_2, I_2, S_2^*, 0) = 0,$ and $V_3(S, I, S^*, I^*) > 0,\ \forall (S, I, S^*, I^*) \neq (S_2, I_2, S_2^*, 0)$. Further we have:

ISSN 2347-4289

$$\frac{dV_3}{dt}=\left(\frac{S-S_2}{S}\right)\frac{dS}{dt}+\left(\frac{I-I_2}{I}\right)\frac{dI}{dt}+\left(\frac{S^*-S_2^*}{S^*}\right)\frac{dS^*}{dt}+\frac{dI^*}{dt}$$

By simplifying this equation we get:

$$\begin{aligned}\frac{dV_3}{dt}=&-\frac{\mu}{S^*}\left(S^*-S_2^*\right)^2-\left(\frac{\beta_3+\mu}{S}\right)(S-S_2)^2\\&+\frac{\beta_3}{I}(S-S_2)(I-I_2)-\left(\frac{\mu+d_1}{I}\right)(I-I_2)^2\\&+\left(\beta_4+\frac{\beta_2 I}{N}\right)S_2^*-(\mu+d_2)I^*+\frac{\beta_1 I^*}{N}\left(S_2-\frac{SI_2}{I}\right)\end{aligned}$$

Therefore, according to condition (16a) it is obtain that:

$$\begin{aligned}\frac{dV_3}{dt}\le&-\frac{\mu}{S^*}\left(S^*-S_2^*\right)^2\\&-\left[\sqrt{\frac{\beta_3+\mu}{S}}(S-S_2)-\sqrt{\frac{\mu+d_1}{I}}(I-I_2)\right]^2\\&+\left(\beta_4+\frac{\beta_2 I}{N}\right)S_2^*-(\mu+d_2)I^*+\frac{\beta_1 I^*}{N}\left(S_2-\frac{SI_2}{I}\right)\end{aligned}$$

Obviously, $\frac{dV_3}{dt}<0$ for every initial points satisfying conditions (16b) and then V_3 is a Lyapunov function provided that conditions (16a)-(16b) hold. Thus E_2 is globally asymptotically stable in the interior of $B(E_2)$, which means that $B(E_2)$ is the basin of attraction and that complete the proof. ■

<u>Theorem (9):</u> Let the endemic equilibrium point E_3 of system (1) is locally asymptotically stable. Then it is globally asymptotically stable provided that:

$$\left[\beta_3 N+\beta_1 I^*\right]^2<\frac{2}{3}\left[N(\beta_3+\mu)+\beta_1 I^*\right]\left[N(\mu+d_1)\right] \quad (17a)$$

$$[\beta_1 S_3]^2<\frac{2}{3}\left[N(\beta_3+\mu)+\beta_1 I^*\right]\left[N(\mu+d_2)\right] \quad (17b)$$

$$\left[\beta_2 S^*\right]^2<\frac{2}{3}\left[N(\mu+d_1)\right]\left[N(\beta_4+\mu)+\beta_2 I_3\right] \quad (17c)$$

$$\left[\beta_1 S_3+\beta_2 S_3^*\right]^2<\frac{4}{9}N^2[\mu+d_1][\mu+d_2] \quad (17d)$$

$$[\beta_4 N+\beta_2 I_3]^2<\frac{2}{3}[N(\beta_4+\mu)+\beta_2 I_3][N(\mu+d_2)] \quad (17e)$$

<u>Proof:</u> Consider the following positive definite function:

$$V_4=\frac{(S-S_3)^2}{2}+\frac{(I-I_3)^2}{2}+\frac{\left(S^*-S_3^*\right)^2}{2}+\frac{\left(I^*-I_3^*\right)^2}{2}$$

Clearly, $V_4:R_+^4\to R$ is a continuously differentiable function such that $V_4(S_3,I_3,S_3^*,I_3^*)=0$ and $V_4(S,I,S^*,I^*)>0,\ \forall(S,I,S^*,I^*)\ne(S_3,I_3,S_3^*,I_3^*)$. Further, we have:

$$\frac{dV_4}{dt}=(S-S_3)\frac{dS}{dt}+(I-I_3)\frac{dI}{dt}+(S^*-S_3^*)\frac{dS^*}{dt}+(I^*-I_3^*)\frac{dI^*}{dt}$$

By simplifying this equation we get:

$$\begin{aligned}\frac{dV_4}{dt}=&-\frac{1}{2}a_{11}(S-S_3)^2+a_{12}(S-S_3)(I-I_3)-\frac{1}{3}a_{22}(I-I_3)^2\\&-\frac{1}{2}a_{11}(S-S_3)^2-a_{14}(S-S_3)\left(I^*-I_3^*\right)-\frac{1}{3}a_{44}\left(I^*-I_3^*\right)^2\\&-\frac{1}{3}a_{22}(I-I_3)^2-a_{23}(I-I_3)\left(S^*-S_3^*\right)-\frac{1}{2}a_{33}\left(S^*-S_3^*\right)^2\\&-\frac{1}{3}a_{22}(I-I_3)^2+a_{24}(I-I_3)\left(I^*-I_3^*\right)-\frac{1}{3}a_{44}\left(I^*-I_3^*\right)^2\\&-\frac{1}{2}a_{33}\left(S^*-S_3^*\right)^2+a_{34}\left(S^*-S_3^*\right)\left(I^*-I_3^*\right)-\frac{1}{3}a_{44}\left(I^*-I_3^*\right)^2\end{aligned}$$

With

$$a_{11}=\left[(\beta_3+\mu)+\frac{\beta_1 I^*}{N}\right];a_{12}=\left[\beta_3+\frac{\beta_1 I^*}{N}\right];a_{22}=[\mu+d_1]$$

$$a_{14}=\left[\frac{\beta_1 S_3}{N}\right];a_{44}=[\mu+d_2];a_{23}=\left[\frac{\beta_2 S^*}{N}\right];a_{33}=\left[(\beta_4+\mu)+\frac{\beta_2 I_3}{N}\right]$$

$$a_{24}=\left[\frac{\beta_1 S_3+\beta_2 S_3^*}{N}\right];a_{34}=\left[\beta_4+\frac{\beta_2 I_3}{N}\right]$$

Therefore, according to the conditions (17a)-(17e) we obtain that:

$$\begin{aligned}\frac{dV_4}{dt}\le&-\frac{1}{N}\left[\sqrt{\frac{N(\beta_3+\mu)+\beta_1 I^*}{2}}(S-S_3)-\sqrt{\frac{N(\mu+d_1)}{3}}(I-I_3)\right]^2\\&-\frac{1}{N}\left[\sqrt{\frac{N(\beta_3+\mu)+\beta_1 I^*}{2}}(S-S_3)+\sqrt{\frac{N(\mu+d_2)}{3}}\left(I^*-I_3^*\right)\right]^2\\&-\frac{1}{N}\left[\sqrt{\frac{N(\mu+d_1)}{3}}(I-I_3)+\sqrt{\frac{N(\beta_4+\mu)+\beta_2 I_3}{2}}\left(S^*-S_3^*\right)\right]^2\\&-\frac{1}{N}\left[\sqrt{\frac{N(\mu+d_1)}{3}}(I-I_3)-\sqrt{\frac{N(\mu+d_2)}{3}}\left(I^*-I_3^*\right)\right]^2\\&-\frac{1}{N}\left[\sqrt{\frac{N(\beta_4+\mu)+\beta_2 I_3}{2}}\left(S^*-S_3^*\right)-\sqrt{\frac{N(\mu+d_2)}{3}}\left(I^*-I_3^*\right)\right]^2\end{aligned}$$

Clearly, $\frac{dV_4}{dt}<0$, and then V_4 is a Lyapunov function provided that the given conditions hold. Therefore, E_3 is globally asymptotically stable. ■

ISSN 2347-4289

6. Local bifurcation analysis of system (1)

In this section, the effect of varying parameter on the dynamical behavior of the system (1) around each equilibrium points is studied. Recall that the existence of non-hyperbolic equilibrium point of the system (1) is the necessary but not sufficient condition for bifurcation to accurse. Therefore, in the following theorems and application to the Sotomayor's theorem for local bifurcation [13], is adapted.

Local bifurcation near (E_0):

Theorem (10): The system (1) at the disease free equilibrium point (E_0) with the parameter $\beta_1^{[0]}=\frac{N^2(\beta_3+\mu)(\beta_4+\mu)(\mu+d_1)(\mu+d_2)}{\beta_2\mu S_0 S_0^*(\beta_3+\beta_4+\mu)}$ has:

1) No saddle-node bifurcation
2) No pitchfork bifurcation
3) A transcritical bifurcation

Proof: According to Jacobian matrix $J(E_0)$ given by equation (12) the system (1) at the disease free equilibrium point (E_0) has zero eigenvalue $(say\ \lambda=0)$ if and only if $\det(J(E_0))=0$, therefore, $\beta_1^{[0]}=\frac{N^2(\beta_3+\mu)(\beta_4+\mu)(\mu+d_1)(\mu+d_2)}{\beta_2\mu S_0 S_0^*(\beta_3+\beta_4+\mu)}$ is taken as a candidate bifurcation parameter, and the Jacobian matrix $J(E_0)$ with $\beta_1=\beta_1^{[0]}$ becomes:

$$\tilde{J}=J_0(\lambda=0)=\begin{bmatrix} -(\beta_3+\mu) & 0 & 0 & -\frac{\beta_1^{[0]}S_0}{N} \\ \beta_3 & -(\mu+d_1) & 0 & \frac{\beta_1^{[0]}S_0}{N} \\ 0 & -\frac{\beta_2 S_0^*}{N} & -(\beta_4+\mu) & 0 \\ 0 & \frac{\beta_2 S_0^*}{N} & \beta_4 & -(\mu+d_2) \end{bmatrix}$$

Further the eigenvector (say $K^{[0]}=(k_1,k_2,k_3,k_4)^T$) corresponding to $\lambda=0$ satisfy the following:

$$\tilde{J}_0 K^{[0]}=\lambda K^{[0]} \quad \text{Then} \quad \tilde{J}_0 K^{[0]}=0$$

From which we get that:

$$-(\beta_3+\mu)k_1-\frac{\beta_1^{[0]}S_0 k_4}{N}=0 \qquad (18a)$$

$$\beta_3 k_1-(\mu+d_1)k_2+\frac{\beta_1^{[0]}S_0 k_4}{N}=0 \qquad (18b)$$

$$-\frac{\beta_2 S_0^* k_2}{N}-(\beta_4+\mu)k_3=0 \qquad (18c)$$

$$\frac{\beta_2 S_0^* k_2}{N}+\beta_4 k_3-(\mu+d_2)k_4=0 \qquad (18d)$$

So by solving the above system of equations we get:

$$k_1=-pk_4;\ k_2=qk_4;\ k_3=-zk_4$$

Where:

$$p=\frac{\beta_1^{[0]}S_0}{N(\beta_3+\mu)}$$
$$q=\frac{\beta_1^{[0]}S_0\mu}{N(\beta_3+\mu)(\mu+d_1)}$$
$$z=\frac{\beta_1^{[0]}\beta_2\mu S_0 S_0^*}{N^2(\beta_3+\mu)(\beta_4+\mu)(\mu+d_1)}$$

Here k_4 be any non zero real number. Thus

$$K^{[0]}=\begin{bmatrix} -pk_4 \\ qk_4 \\ -zk_4 \\ k_4 \end{bmatrix}$$

Similarly the eigenvector $W^{[0]}=(w_1\ , w_2\ , w_3\ , w_4)^T$ that corresponding to $\lambda=0$ of $\tilde{J}_0^T$ can be written:

$$\tilde{J}_0^T\cdot W^{[0]}=\begin{bmatrix} -(\beta_3+\mu) & \beta_3 & 0 & 0 \\ 0 & -(\mu+d_1) & -\frac{\beta_2 S_0^*}{N} & \frac{\beta_2 S_0^*}{N} \\ 0 & 0 & -(\beta_4+\mu) & \beta_4 \\ -\frac{\beta_1^{[0]}S_0}{N} & \frac{\beta_1^{[0]}S_0}{N} & 0 & -(\mu+d_2) \end{bmatrix}\cdot\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}=0$$

This gives:

$$W^{[0]}=\begin{bmatrix} xw_4 \\ mw_4 \\ yw_4 \\ w_4 \end{bmatrix}$$

Where

$$x=\frac{\beta_2\beta_3\beta_4\mu S_0^*}{N(\beta_3+\mu)(\beta_4+\mu)(\mu+d_1)}$$
$$m=\frac{\beta_2\beta_4\mu S_0^*}{N(\beta_4+\mu)(\mu+d_1)}$$
$$y=\frac{\beta_4}{(\beta_4+\mu)}$$

Here w_4 be any non zero real number. Now, rewrite system (1) in a vector form as:

ISSN 2347-4289

$$\frac{dX}{dt} = f(X)$$

Where $X = \left(S, I, S^*, I^*\right)^T$ and $f = (f_1, f_2, f_3, f_4)^T$ with $f_i, i = 1,2,3,4$ are given in system (1), and then determine $\frac{df}{d\beta_1} = f_{\beta_1}$ we get that:

$$f_{\beta_1} = \begin{bmatrix} -\frac{SI^*}{N} \\ \frac{SI^*}{N} \\ 0 \\ 0 \end{bmatrix} \quad \text{Then} \quad f_{\beta_1}(E_0, \beta_1^{[0]}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore:

$$W^{[0]T} \cdot f_{\beta_1}(E_0, \beta_1^{[0]}) = 0$$

Consequently, according to Sotomayor theorem [13] the system (1) has no saddle-node bifurcation near E_0 at $\beta_1^{[0]}$. Now in order to investigate the accruing of other types of bifurcation, the derivative of f_{β_1} with respect to vector X, say $Df_{\beta_1}(E_0, \beta_1^{[0]})$, is computed

$$Df_{\beta_1}(E_0, \beta_1^{[0]}) = \begin{bmatrix} 0 & 0 & 0 & -\frac{S_0}{N} \\ 0 & 0 & 0 & \frac{S_0}{N} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$$W^{[0]T} \cdot \left[Df_{\beta_1}(E_0, \beta_1^{[0]}) \cdot K^{[0]}\right] = -\frac{S_0 k_4 w_4}{N}(x - m) \neq 0$$

Again, according to Sotomayor theorem, if in addition to the above, the following holds

$$W^{[0]T} \cdot \left[D^2 f(E_0, \beta_1^{[0]}) \cdot (K^{[0]}, K^{[0]})\right] \neq 0$$

Here $Df(E_0, \beta_1^{[0]})$ is the Jacobian matrix at E_0 and $\beta_1^{[0]}$, then the system (1) possesses a transcritical bifurcation but no pitch-fork bifurcation can occur. Now since we have that:

$$\left[D^2 f(E_0, \beta_1^{[0]}) \cdot (K^{[0]}, K^{[0]})\right] = \begin{bmatrix} \frac{2\beta_1 p k_4^2}{N} \\ -\frac{2\beta_1 p k_4^2}{N} \\ \frac{2\beta_2 q z k_4^2}{N} \\ -\frac{2\beta_2 q z k_4^2}{N} \end{bmatrix}$$

Therefore:

$$W^{[0]T} \cdot \left[D^2 f(E_0, \beta_1^{[0]}) \cdot (K^{[0]}, K^{[0]})\right] = \frac{2\beta_1 p k_4^2 w_4}{N}(x - m) + \frac{2\beta_2 q z k_4^2 w_4}{N}(y - 1) \neq 0$$

Then the system (1) has a transcritical bifurcation at E_0 when the parameter β_1 passes through the bifurcation value $\beta_1^{[0]}$.

Local bifurcation near (E_1) :

<u>Theorem (11):</u> The system (1) at the male's disease free equilibrium point (E_1) with the parameter $\beta_2^{[1]} = \frac{N^3(\beta_4 + \mu)\left(N\beta_3 + \beta_1 I_1^* + N\mu\right)(\mu + d_1)(\mu + d_2)}{N\beta_1 \mu S_1 S_1^*\left(\beta_3 N + \beta_1 I_1^*\right)}$ has:

1) No saddle-node bifurcation
2) No pitchfork bifurcation
3) A transcritical bifurcation

<u>Proof:</u> According to Jacobian matrix $J(E_1)$ given by equation (13) the system (1) at the male's disease free equilibrium point (E_1) has zero eigenvalue $\left(say\ \tilde{\lambda} = 0\right)$ if and only if $\det\left(J(E_1)\right) = 0$, therefore, $\beta_2^{[1]} = \frac{N^3(\beta_4 + \mu)\left(N\beta_3 + \beta_1 I_1^* + N\mu\right)(\mu + d_1)(\mu + d_2)}{N\beta_1 \mu S_1 S_1^*\left(\beta_3 N + \beta_1 I_1^*\right)}$ is taken as a candidate bifurcation parameter, and the Jacobian matrix $J(E_1)$ with $\beta_2 = \beta_2^{[1]}$ becomes:

$$\bar{J} = J_1\left(\tilde{\lambda} = 0\right) = \begin{bmatrix} -(\beta_3 + \frac{\beta_1 I_1^*}{N} + \mu) & 0 & 0 & -\frac{\beta_1 S_1}{N} \\ \beta_3 + \frac{\beta_1 I_1^*}{N} & -(\mu + d_1) & 0 & \frac{\beta_1 S_1}{N} \\ 0 & -\frac{\beta_2^{[1]} S_1^*}{N} & -(\beta_4 + \mu) & 0 \\ 0 & \frac{\beta_2^{[1]} S_1^*}{N} & \beta_4 & -(\mu + d_2) \end{bmatrix}$$

Further the eigenvector (say $K^{[1]} = \left(k_1, k_2, k_3, k_4\right)^T$) corresponding to $\tilde{\lambda} = 0$ satisfy the following:

$$\bar{J}_1 K^{[1]} = \tilde{\lambda} K^{[1]} \quad \text{Then} \quad \bar{J}_1 K^{[1]} = 0$$

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From which we get that:

$$-\left(\beta_3+\frac{\beta_1 I_1^*}{N}+\mu\right)k_1-\frac{\beta_1 S_1 k_4}{N}=0 \qquad (19a)$$

$$\left(\beta_3+\frac{\beta_1 I_1^*}{N}\right)k_1-(\mu+d_1)k_2+\frac{\beta_1 S_1 k_4}{N}=0 \qquad (19b)$$

$$-\frac{\beta_2^{[1]} S_1^* k_2}{N}-\left(\beta_4+\mu\right)k_3=0 \qquad (19c)$$

$$\frac{\beta_2^{[1]} S_1^* k_2}{N}+\beta_4 k_3-\left(\mu+d_2\right)k_4=0 \qquad (19d)$$

So by solving the above system of equations we get:

$$k_1=-pk_4\,;\; k_2=qk_4\,;\; k_3=-zk_4$$

Where:

$$p=\frac{\beta_1 S_1}{N\beta_3+\beta_1 I_1^*+N\mu}$$
$$q=\frac{\beta_1 S_1 \mu}{\left(\mu+d_1\right)\left(N\beta_3+\beta_1 I_1^*+N\mu\right)}$$
$$z=\frac{\beta_2^{[1]}\beta_1 \mu S_1 S_1^*}{N\left(\beta_4+\mu\right)\left(\mu+d_1\right)\left(N\beta_3+\beta_1 I_1^*+N\mu\right)}$$

Here k_4 be any non zero real number. Thus

$$K^{[1]}=\begin{bmatrix}-pk_4\\ qk_4\\ -zk_4\\ k_4\end{bmatrix}$$

Similarly the eigenvector $W^{[1]}=\left(w_1\,,\,w_2\,,\,w_3\,,\,w_4\right)^T$ that corresponding to $\tilde{\lambda}=0$ of $\bar{J}_1^T$ can be written:

$$\bar{J}_1^T\cdot W^{[1]}=\begin{bmatrix}-\left(\beta_3+\frac{\beta_1 I_1^*}{N}+\mu\right) & \beta_3+\frac{\beta_1 I_1^*}{N} & 0 & 0\\ 0 & -\left(\mu+d_1\right) & -\frac{\beta_2^{[1]}S_1^*}{N} & \frac{\beta_2^{[1]}S_1^*}{N}\\ 0 & 0 & -\left(\beta_4+\mu\right) & \beta_4\\ -\frac{\beta_1 S_1}{N} & \frac{\beta_1 S_1}{N} & 0 & -\left(\mu+d_2\right)\end{bmatrix}\cdot\begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{bmatrix}=0$$

This gives:

$$W^{[1]}=\begin{bmatrix}xw_4\\ mw_4\\ yw_4\\ w_4\end{bmatrix}$$

Where

$$x=\frac{\beta_2^{[1]}\mu S_1^*\left(\beta_3 N+\beta_1 I_1^*\right)}{N\left(\beta_4+\mu\right)\left(\mu+d_1\right)\left(N\beta_3+\beta_1 I_1^*+N\mu\right)}$$
$$m=\frac{-\beta_2^{[1]}\mu S_1^*}{N\left(\beta_4+\mu\right)\left(\mu+d_1\right)}$$
$$y=\frac{\beta_4}{\left(\beta_4+\mu\right)}$$

Here w_4 be any non zero real number. Now, rewrite system (1) in a vector form as:

$$\frac{dX}{dt}=f(X)$$

Where $X=\left(S,I,S^*,I^*\right)^T$ and $f=(f_1,f_2,f_3,f_4)^T$ with $f_i, i=1,2,3,4$ are given in system (1), and then determine $\frac{df}{d\beta_2}=f_{\beta_2}$ we get that:

$$f_{\beta_1}=\begin{bmatrix}0\\ 0\\ -\frac{S^* I}{N}\\ \frac{S^* I}{N}\end{bmatrix} \quad \text{then} \quad f_{\beta_2}(E_1,\beta_2^{[1]})=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

Therefore:

$$W^{[1]^T}\cdot f_{\beta_2}(E_1,\beta_2^{[1]})=0$$

Consequently, according to Sotomayor theorem [13] the system (1) has no saddle-node bifurcation near E_1 at $\beta_2^{[1]}$. Now in order to investigate the accruing of other types of bifurcation, the derivative of f_{β_2} with respect to vector X, say $Df_{\beta_2}(E_1,\beta_2^{[1]})$, is computed

$$Df_{\beta_2}(E_1,\beta_2^{[1]})=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -\frac{S^*}{N} & 0 & 0\\ 0 & \frac{S^*}{N} & 0 & 0\end{bmatrix}$$

So

$$W^{[1]^T}\cdot\left[Df_{\beta_2}(E_1,\beta_2^{[1]})\cdot K^{[1]}\right]=-\frac{S^* q k_4 w_4}{N}\left(y-1\right)\neq 0$$

Again, according to Sotomayor theorem, if in addition to the above, the following holds

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$$W^{[1]^T}\cdot\left[D^2 f(E_1,\beta_2^{[1]})\cdot(K^{[1]},K^{[1]})\right]\neq 0$$

Here $Df(E_1,\beta_2^{[1]})$ is the Jacobian matrix at E_1 and $\beta_2^{[1]}$, then the system (1) possesses a transcritical bifurcation but no pitch-fork bifurcation can occur. Now since we have that:

$$\left[D^2 f(E_1,\beta_2^{[1]})\cdot(K^{[1]},K^{[1]})\right]=\begin{bmatrix}\frac{2\beta_1 pk_4^2}{N}\\ -\frac{2\beta_1 pk_4^2}{N}\\ \frac{2\beta_2 qzk_4^2}{N}\\ -\frac{2\beta_2 qzk_4^2}{N}\end{bmatrix}$$

Therefore:

$$W^{[0]^T}\cdot\left[D^2 f(E_0,\beta_1^{[0]})\cdot(K^{[0]},K^{[0]})\right]$$
$$=\frac{2\beta_1 pk_4^2 w_4}{N}\left(x-m\right)+\frac{2\beta_2 qzk_4^2 w_4}{N}\left(y-1\right)\neq 0$$

Then the system (1) has a transcritical bifurcation at E_1 when the parameter β_2 passes through the bifurcation value $\beta_2^{[1]}$. ■

Local bifurcation near (E_2) :

Theorem (12): The system (1) at the male's disease free equilibrium point (E_2) with the parameter $\beta_1^{[2]}=\frac{N^3(\beta_3+\mu)(N\beta_4+\beta_2 I_2+N\mu)(\mu+d_1)(\mu+d_2)}{N\beta_2\mu S_2 S_2^*[\beta_2 I_2+N(1+\beta_4+\mu)]}$ has:

1) No saddle-node bifurcation
2) No pitchfork bifurcation
3) A transcritical bifurcation

Proof: According to Jacobian matrix $J(E_2)$ given by equation (14) the system (1) at the male's disease free equilibrium point (E_2) has zero eigenvalue $\left(say\ \bar{\lambda}=0\right)$ if and only if $\det\left(J(E_2)\right)=0$, therefore, $\beta_1^{[2]}=\frac{N^3(\beta_3+\mu)(N\beta_4+\beta_2 I_2+N\mu)(\mu+d_1)(\mu+d_2)}{N\beta_2\mu S_2 S_2^*[\beta_2 I_2+N(1+\beta_4+\mu)]}$ is taken as a candidate bifurcation parameter, and the Jacobian matrix $J(E_2)$ with $\beta_1=\beta_1^{[2]}$ becomes:

$$\hat{J}=J_2\left(\bar{\lambda}=0\right)$$
$$=\begin{bmatrix}-(\beta_3+\mu) & 0 & 0 & -\frac{\beta_1^{[2]}S_2}{N}\\ \beta_3 & -(\mu+d_1) & 0 & \frac{\beta_1^{[2]}S_2}{N}\\ 0 & -\frac{\beta_2 S_2^*}{N} & -(\beta_4+\frac{\beta_2 I_2}{N}+\mu) & 0\\ 0 & \frac{\beta_2 S_2^*}{N} & \beta_4+\frac{\beta_2 I_2}{N} & -(\mu+d_2)\end{bmatrix}$$

Further the eigenvector (say $K^{[2]}=\left(k_1,k_2,k_3,k_4\right)^T$) corresponding to $\bar{\lambda}=0$ satisfy the following:

$$\hat{J}_2K^{[2]}=\bar{\lambda}K^{[2]}\ \text{Then}\ \hat{J}_2K^{[2]}=0$$

From which we get that:

$$-\left(\beta_3+\mu\right)k_1-\frac{\beta_1^{[2]}S_2k_4}{N}=0 \quad (20a)$$

$$\beta_3k_1-(\mu+d_1)k_2+\frac{\beta_1^{[2]}S_2k_4}{N}=0 \quad (20b)$$

$$-\frac{\beta_2S_2^*k_2}{N}-\left(\beta_4+\frac{\beta_2I_2}{N}+\mu\right)k_3=0 \quad (20c)$$

$$\frac{\beta_2S_2^*k_2}{N}+\left(\beta_4+\frac{\beta_2I_2}{N}\right)k_3-\left(\mu+d_2\right)k_4=0 \quad (20d)$$

So by solving the above system of equations we get:

$$k_1=-pk_4;\ k_2=qk_4;\ k_3=-zk_4$$

Where:

$$p=\frac{\beta_1^{[2]}S_2}{N\left(\beta_3+\mu\right)}$$
$$q=\frac{\beta_1^{[2]}S_2\mu}{N\left(\beta_3+\mu\right)\left(\mu+d_1\right)}$$
$$z=\frac{\beta_1^{[2]}\beta_2\mu S_2S_2^*}{N\left(\beta_3+\mu\right)\left(N\beta_4+\beta_2I_2+N\mu\right)\left(\mu+d_1\right)}$$

Here k_4 be any non zero real number. Thus

$$K^{[2]}=\begin{bmatrix}-pk_4\\ qk_4\\ -zk_4\\ k_4\end{bmatrix}$$

Similarly the eigenvector $W^{[2]}=\left(w_1\ ,\ w_2\ ,\ w_3\ ,\ w_4\right)^T$ that corresponding to $\bar{\lambda}=0$ of $\hat{J}_2^T$ can be written:

ISSN 2347-4289

$$\hat{j}_2^T \cdot W^{[2]} = \begin{bmatrix} -(\beta_3+\mu) & \beta_3 & 0 & 0 \\ 0 & -(\mu+d_1) & -\frac{\beta_2 S_2^*}{N} & \frac{\beta_2 S_2^*}{N} \\ 0 & 0 & -\left(\beta_4+\frac{\beta_2 I_2}{N}+\mu\right) & \beta_4+\frac{\beta_2 I_2}{N} \\ -\frac{\beta_1^{[2]} S_2}{N} & \frac{\beta_1^{[2]} S_2}{N} & 0 & -(\mu+d_2) \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = 0$$

This gives:

$$W^{[2]} = \begin{bmatrix} xw_4 \\ mw_4 \\ yw_4 \\ w_4 \end{bmatrix}$$

Where

$$x = \frac{\beta_2 \beta_3 \mu S_2^*}{(\beta_3+\mu)(N\beta_4+\beta_2 I_2+\mu N)(\mu+d_1)}$$

$$m = \frac{\beta_2 \mu S_2^*}{(N\beta_4+\beta_2 I_2+N\mu)(\mu+d_1)}$$

$$y = \frac{N\beta_4+\beta_2 I_2}{N\beta_4+\beta_2 I_2+N\mu}$$

Here w_4 be any non zero real number. Now, rewrite system (1) in a vector form as:

$$\frac{dX}{dt} = f(X)$$

Where $X = \left(S, I, S^*, I^*\right)^T$ and $f = (f_1, f_2, f_3, f_4)^T$ with $f_i, i = 1,2,3,4$ are given in system (1), and then determine $\frac{df}{d\beta_1} = f_{\beta_1}$ we get that:

$$f_{\beta_1} = \begin{bmatrix} -\frac{SI^*}{N} \\ \frac{SI^*}{N} \\ 0 \\ 0 \end{bmatrix} \quad \text{Then} \quad f_{\beta_1}(E_2, \beta_1^{[2]}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore:

$$W^{[2]T} \cdot f_{\beta_1}(E_2, \beta_1^{[2]}) = 0$$

Consequently, according to Sotomayor theorem [13] the system (1) has no saddle-node bifurcation near E_2 at $\beta_1^{[2]}$. Now in order to investigate the accruing of other types of bifurcation, the derivative of f_{β_1} with respect to vector X, say $Df_{\beta_1}(E_2, \beta_1^{[2]})$, is computed

$$Df_{\beta_1}(E_2, \beta_1^{[2]}) = \begin{bmatrix} 0 & 0 & 0 & -\frac{S_2}{N} \\ 0 & 0 & 0 & \frac{S_2}{N} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$$W^{[2]T} \cdot \left[Df_{\beta_1}(E_2, \beta_1^{[2]}) \cdot K^{[2]}\right] = -\frac{S_2 k_4 w_4}{N}(x-m) \neq 0$$

Again, according to Sotomayor theorem, if in addition to the above, the following holds

$$W^{[2]T} \cdot \left[D^2 f(E_2, \beta_1^{[2]}) \cdot (K^{[2]}, K^{[2]})\right] \neq 0$$

Here $Df(E_2, \beta_1^{[2]})$ is the Jacobian matrix at E_2 and $\beta_1^{[2]}$, then the system (1) possesses a transcritical bifurcation but no pitch-fork bifurcation can occur. Now since we have that:

$$\left[D^2 f(E_2, \beta_1^{[2]}) \cdot (K^{[2]}, K^{[2]})\right] = \begin{bmatrix} \frac{2\beta_1 p k_4^2}{N} \\ -\frac{2\beta_1 p k_4^2}{N} \\ \frac{2\beta_2 q z k_4^2}{N} \\ -\frac{2\beta_2 q z k_4^2}{N} \end{bmatrix}$$

Therefore:

$$W^{[2]T} \cdot \left[D^2 f(E_2, \beta_1^{[2]}) \cdot (K^{[2]}, K^{[2]})\right] = \frac{2\beta_1 p k_4^2 w_4}{N}(x-m) + \frac{2\beta_2 q z k_4^2 w_4}{N}(y-1) \neq 0$$

Then the system (1) has a transcritical bifurcation at E_2 when the parameter β_1 passes through the bifurcation value $\beta_1^{[2]}$. ■

Local bifurcation near (E_3) :

The occurrence of local bifurcation near the endemic equilibrium point (E_3) of system (1) is also studied. Not that, it is well known that the necessary condition of the system (1) to have a local bifurcation (saddle-node, transcritical and pitchfork bifurcation) around (E_3) at a specific parameter are given by, $\det(J(E_3)) = 0$, where

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$J(E_3)$ represent the Jacobian matrix of the system (1). Now since the condition that guarantee to make $\det(J(E_3))=0$ does not exist. Hence there is no possibility of occurrence of local bifurcation.

7. Numerical analysis of systems (1):

In this section, the global dynamic of system (1) is studied numerically. The objectives of this study are confirming our obtained analytical results and understand the effects of contact and existence of the external sources for disease on the dynamic of SIS^*I^* epidemic model. Consequently, the system (1) is solved numerically for different sets of initial conditions and for different sets of parameters. It is observed that, for the following set of hypothetical parameters given equation (21) with $\beta_i=0, i=1,2,3,4$ that satisfies stability condition (10) of disease free equilibrium point, system (1) has a globally asymptotically stable endemic equilibrium point as shown in following figure.

$$\Lambda_1=500,\ \Lambda_2=400,\ \beta_1=0.0002,\ \beta_2=0.002,\ \beta_3=0.0001, \beta_4=0.0003,\ \mu=0.3,\ d_1=0.5,\ d_2=0.4 \qquad (21)$$

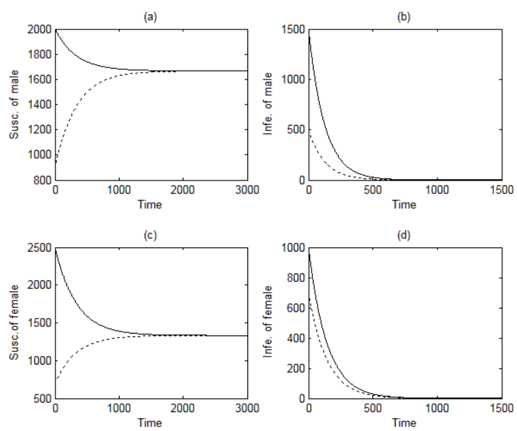


Figure (2): Time series of the solution of system (1). (a) trajectories of S , (b) trajectories of I, (c) trajectories of S^* and (d) trajectories of I^*. The solid line refers to the trajectory started at (2000, 1500, 2500, 1000), while the dotted line refers to the trajectory started at (900, 500, 700, 700).

Obviously, Figure (2) shows clearly the convergence of system (1) to the disease free equilibrium point $E_0=(1667,0,1333,0)$ asymptotically from two different initial points. However, for the data given equation (21) with $\beta_i=0, i=1,3$. The trajectories of system (1) starting from different sets of initial data are drawn in Figure. (3).

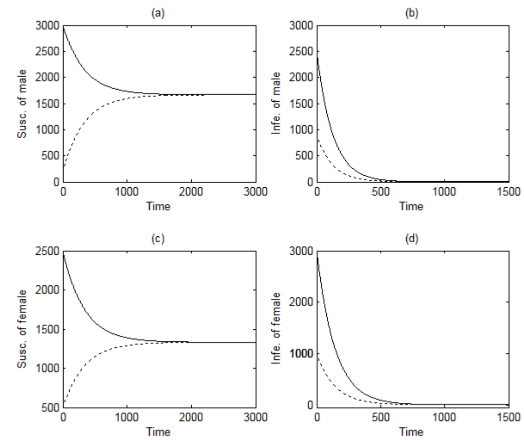


Figure (3): Time series of the solution of system (1). (a) trajectories of S , (b) trajectories of I, (c) trajectories of S^* and (d) trajectories of I^*. The solid line refers to the trajectory started at (3000, 2500, 2500, 3000), while the dotted line refers to the trajectory started at (200, 900, 500, 1000).

Obviously, Figure (3) shows clearly the convergence of system (1) to the infected free of male equilibrium point $E_1=(1667,0,1332,0.57)$ asymptotically from two different initial points. However, for the data given equation (21) with $\beta_i=0, i=2,4$. The trajectories of system (1) starting from different sets of initial data are drawn in Figure. (4).

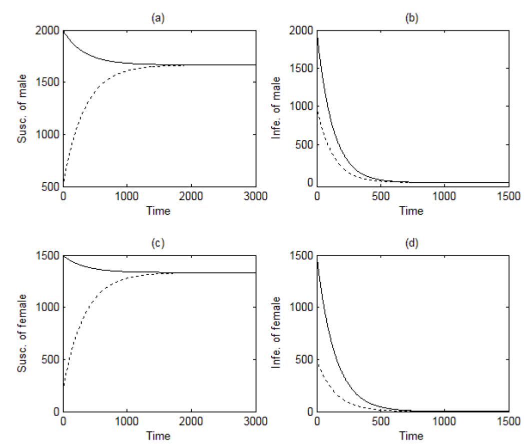


Figure (4): Time series of the solution of system (1). (a) trajectories of S , (b) trajectories of I, (c) trajectories of S^* and (d) trajectories of I^*. The solid line refers to the trajectory started at (2000, 2000, 1500, 1500), while the dotted line refers to the trajectory started at (500, 1000, 200, 500).

Obviously, Figure (4) shows clearly the convergence of system (1) to the infected free of female equilibrium point $E_2=(1666,0.20,1333,0)$ asymptotically from two different initial points. Now, for the data given equation (21), the trajectories of system (1) starting from different sets of initial data are drawn in Figure. (5).

ISSN 2347-4289

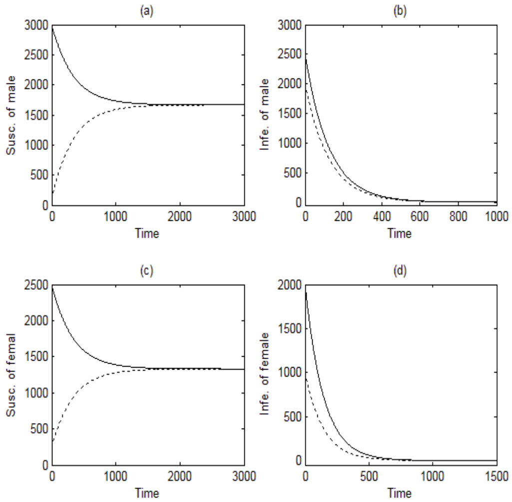
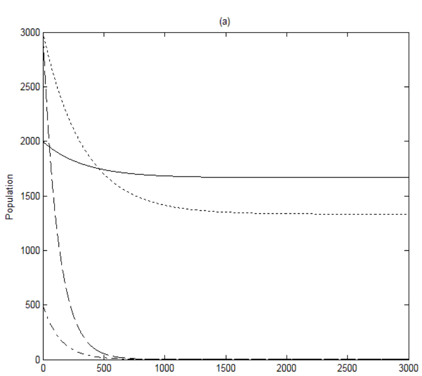
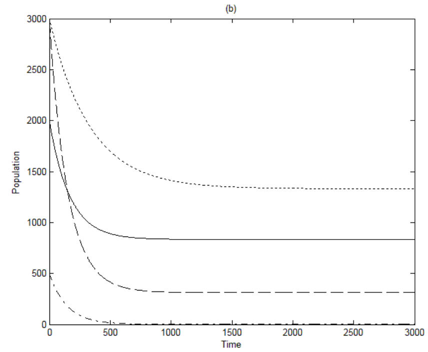


Figure (5): Time series of the solution of system (1). (a) trajectories of S , (b) trajectories of I, (c) trajectories of S^* and (d) trajectories of I^*. The solid line refers to the trajectory started at (3000, 2500, 2500, 2000), while the dotted line refers to the trajectory started at (100, 2000, 250, 1000).

Obviously, Figure (5) shows clearly the convergence of system (1) to the endemic equilibrium point $E_3 = (1666, 0.20, 1332, 0.57)$ asymptotically from two different initial points. Now the effect of increasing the incidence rate of disease resulting by external sources of susceptible of males on the dynamics of system (1) is studied by solving the system numerically for the parameters values $\beta_3 = 0.0001$, 0.3, 0.5 respectively, keeping other parameters fixed as given in equation (21), and then the trajectories of system (1) are drawn in Figures (6a)-(6c) respectively. Note that, in the next figures (6-9), we will use the following representations: Solid line for describing trajectory of S; dashed line for describing trajectory of I; dot line for describing trajectory of S^*; dash dot line for describing trajectory of I^* and starting at (2000, 3000, 3000, 500).





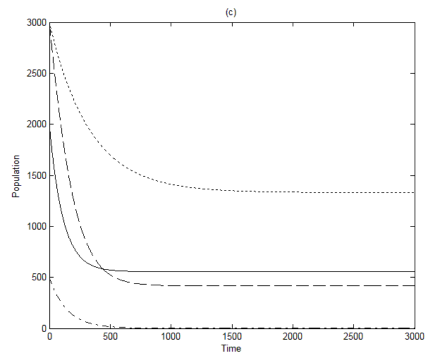


Figure (6): Time series of the solution of system (1). (a) for $\beta_3 = 0.0001$, (b) for $\beta_3 = 0.3$, (c) for $\beta_3 = 0.5$.

According to Figure (6), as the incidence rate of disease resulting by external source increases (through increasing β_3), then the trajectory of system (1) approaches asymptotically to the endemic equilibrium point. In fact as β_3 increases it is observed that the number of susceptible of (males and females) decrease and the number of infected of (males and females) individuals increases. Similar results are obtained, as those shown in case of increasing β_3, in case of increasing the incidence rate of disease resulting by contact between (susceptible of males and infected of females), that is means increasing β_1 and keeping other parameters fixed as given in (21). The effect of increasing the incidence rate of disease resulting from external sources of females on the dynamics of system (1) is studied by solving the system numerically for the parameters values $\beta_4 = 0.0001$, 0.2, 0.7 respectively, keeping other parameters fixed as given in equation (21), and then the trajectories of system (1) are drawn in Figures (7a)-(7c) respectively.

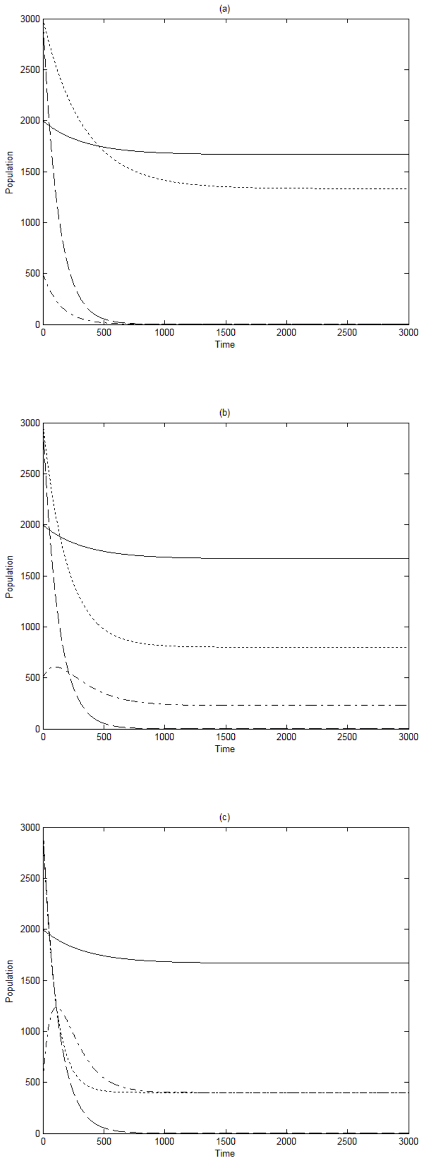


Figure (7): Time series of the solution of system (1). (a) for $\beta_4 = 0.0001$, (b) for $\beta_4 = 0.2$, (c) for $\beta_4 = 0.7$.

According to Figure (7), as the incidence rate of disease resulting from external sources of females increases then the trajectory of system (1) approaches asymptotically to the endemic equilibrium point. In fact as β_4 increases it is observed that the number of susceptible of (males and females) decrease and the number of infected of (males and females) individuals increases. Similar results are obtained, as those shown in case of increasing β_4 , in case of increasing the incidence rate of disease resulting by contact between (susceptible of females and infected of males), that is means increasing β_2 and keeping other parameters fixed as given in (21). In the following, system (1) is solved numerically for the following values of disease related death rate of infected of males $d_1 = 0.5$, 3 , 5 keeping other parameters fixed as given in equation (21), and then the trajectories of system (1) are drawn in Figures (8a)-(8c) respectively.

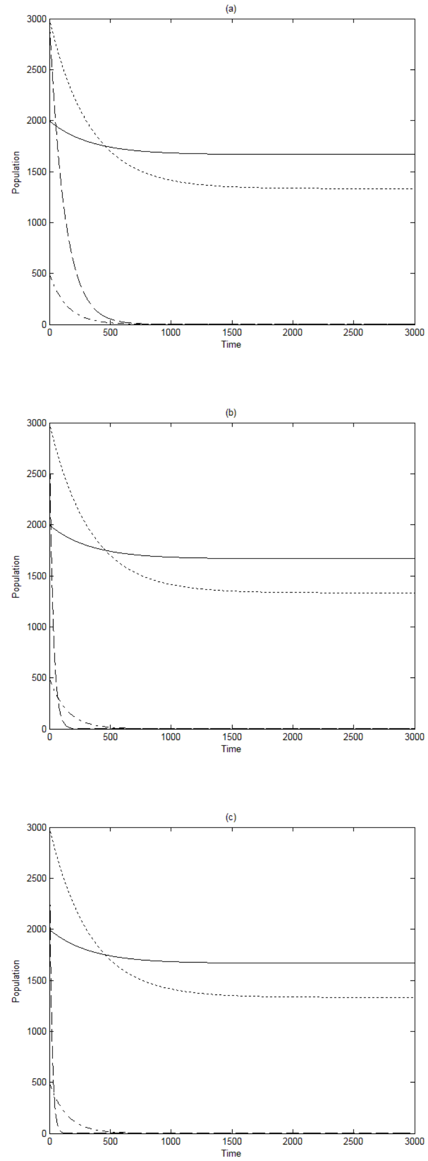


Figure (8): Time series of the solution of system (1). (a) for $d_1 = 0.5$, (b) for $d_1 = 3$, (c) for $d_1 = 5$.

Obviously from these figures, as the disease related rate of males increases the trajectory of system (1) approaches asymptotically to the endemic equilibrium point and the number of susceptible individuals of (males and females) increase and decreasing in the numbers of the infected individuals of (males and females). Similar results are obtained, as those shown in case of increasing d_1 , in case of increasing the disease related rate of females, that is means increasing d_2 and keeping other parameters fixed as given in (21).

8. Conclusion and discussion:

In this paper, we proposed and analyzed an epidemiological model that described the dynamical behavior of an epidemic model, where the infectious disease transmitted directly from external sources as well as through contact between them. The model

ISSN 2347-4289

included fore non-linear autonomous differential equations that describe the dynamics of fore different populations namely susceptible of males (S), infected of males (I), susceptible of female (S^*), infected of females (I^*). The boundedness of system (1) has been discussed. The conditions for existence, stability and bifurcation for each equilibrium points are obtained. Further, it is observed that the disease free equilibrium point (E_0) exists when $I = I^* = 0$ and locally stable if and only if the condition (10) holds, while it is globally stable if and only if the condition (14) holds, while the system (1) has transcritical bifurcation near (E_0) if the condition (18) holds. The disease free for males equilibrium point (E_1) exists when $I = 0$ and locally stable if and only if the conditions (11) hold and it is globally stable if and only if the conditions (15a)-(15b) hold, in addition the system (1) near (E_1) has transcritical bifurcation if the condition (19) holds. The disease free for females equilibrium point (E_2) exists when $I^* = 0$ and locally stable if the conditions (12) hold, while it is globally stable if and only if the conditions (16a)-(16b) hold, in addition the system (1) near (E_2) has transcritical bifurcation if the condition (20) holds. Moreover, the endemic equilibrium point (E_3) of system (1) exists if and only if condition $(D_2 > 0)$ holds, and locally stable if and only if the conditions (13) hold, while it is globally asymptotically stable if and only if the conditions (17a)-(17e) hold. In fact, the system (1) has no any type of bifurcation (saddle-node, pichforck, and transcritical bifurcation). Finally, to understand the effect of varying each parameter on the global system (1) and confirm our above analytical results, the system (1) has been solved numerically for different sets of initial points and different sets of parameters given by equation (21), and the following observations are made:

1. The system (1) do not has periodic dynamic, instead it they approach either to the disease free equilibrium point or else to endemic equilibrium point.
2. As the incidence rate of disease (external incidence rate or contact incidence rate) increase, the asymptotic behavior of the systems (1) transfer from approaching to disease free equilibrium point to the endemic equilibrium point.
3. As the disease related death rate in the systems (1) increase then the solution in the system will be transfer from stability at endemic equilibrium point to stability at disease free equilibrium point.

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