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Neumann-Type A Posteriori Error Estimation For Steady Convection-Diffusion Equation

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ABSTRACT: We consider a steady linear convection – diffusion equation in 2D, present the standard Galerkin (GK) approximation and the Streamline-Diffusion Finite Element Method (SDFEM) and give an analysis of a posteriori error estimator based on solving a local Neumann problem. The estimator gives global upper and local lower bounds on the error measured in the H^1 semi-norm. Our numerical results from GK and SD approximations show that the global effectivity indices deteriorate in rates O(Pe_K) and $O(\sqrt{Pe_K})$ as $Pe_K \to \infty$, respectively i.e., the estimator is over-estimated the error locally within a boundary layer which is not resolved by uniform grid refinement.

Keywords: Convection-diffusion equation, GK Method, SDFEM, a Posteriori error estimator, Effectivity Indices

1 INTRODUCTION

It is well known that the standard Galerkin discretization of convection-diffusion equation yields inaccurate, oscillatory solutions near boundary layers in convection dominated flows and, if the diffusion parameter ϵ is decreased without proportional reduction of the discretization mesh size, then these inaccuracies propagate into regions where the solution is smooth [19]. The streamline-upwind Petrov-Galerkin (SUPG) method [14], [15] or streamline-diffusion finite element method (SDFEM) [18] is designed to overcome these problems by introducing a small amount of artificial diffusion in the direction of streamlines. The numerical solution obtained from the SDFEM has the desirable property that the accuracy in regions where the exact solution is smooth will not be degraded as a result of discontinuities and layers in the exact solution [26], [20]. However, the numerical solution obtained from the SDFEM can be oscillatory in regions where there are layers. To obtain an accurate finite element solution on a given mesh, usually a so-called quasi-uniform or isotropic mesh is desirable [6]. One commom technique to increase the accuracy of the finite element solution is mesh refinement, the so-called h-method. For a refinement procedure to succeed, reliable and efficient a posterori error estimators are needed. For the reliability and efficiency of a posteriori error estimators, a standard measure is the so-called effectivity index, defined as $X_\eta = estimated\ error\ / true\ error$. An estimator is called asymptotically exact if its effectivity index converges to 1 when the mesh size approaches 0. If the effectivity index is much smaller than 1, the estimator is under- estimating the error. On the other hand, if the effectivity index is much greater than 1, the estimator is over-estimating the error. If the estimator does not under-estimate or over-estimate the error globally, then the estimator is reliable, meaning the error on the global domain can be properly controlled by the estimator. If the estimator does not under-estimate or over-estimate the error locally, then the error estimator is efficient, meaning the estimator is able to pinpoint exactly where the error is large and where the error is small. For two-dimensional problems, several estimators have been shown to be asymptotically exact when used on uniform meshes provided the solution of the problem is smooth enough [4], [7], [8]. Estimators based on solving a local Neumann problem, so-called Neumann-type estimators, were first given by [5]. These estimators have been studied by many researchers such as [2], [11], [21], [24], [27], [29]. Our aim is to analyze the reliability of the error estimator proposed by Kay and Silvester [24] which is an extension of the work of Verfurth [28]. In their work, they modify the well-known Bank and Weiser estimator [5], and using the idea of Ainsworth & Oden [1], they solve a local (element) Poisson problem, over a suitably chosen (higher order) approximation space with data from interior residuals and flux jumps along element edges. An outline of this paper is as follows. In Section 2, we present the variational formulation and its finite element discretization of the steady convection-diffusion equation. In Section 3, we discuss the theoretical analysis of the Numann-type a posreriori estimator. In Section 4, we present the numerical results and the effectivity indices for a posteriori estimator and finally, we draw a conclusion.

2 LINEAR CONVECTION-DIFFUSION EQUATION

We consider the following steady, linear convection-diffusion equation

$$-\epsilon\nabla^2 u + \mathbf{b}.\nabla u = f \text{ in } \Omega, \quad (2.1a)$$
$$u = g \text{ on } \Gamma_D, \quad (2.1b)$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N, \quad (2.1c)$$

where $\Omega \subset \mathrm{IR}^2$ is a bounded polygonal domain with Lipschitz boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. We are interested in the convection dominated case and assume that

(A.1) $0 \le \epsilon \ll 1$,
(A.2) $\nabla.\boldsymbol{b} = 0$,
(A.3) $\|\boldsymbol{b}\|_{\infty,\Omega} = 1$,
(A.4) $\boldsymbol{b}.n \ge 0\ on\ \Gamma_N$.

The L_2 norm and the H^1 semi-norm, also called energy norm are defined as

$\|u\|_{H^0(\Omega)} = \|u\|_{L_2(\Omega)} = \left(\int_\Omega u^2 d\Omega\right)^{1/2}$ and

$|u|_{H^1(\Omega)} = \|\nabla u\|_{L_2(\Omega)} = \left(\int_\Omega \sum_{i=1}^d (D_i^1 u)^2\right)^{1/2}$

$\forall\, u \in H^1(\Omega)$, respectively. We shall denote the above norm and semi-norm by the following convention $\|.\|_{k,\Omega} = \|u\|_{H^k(\Omega)}$ and $\|\nabla.\|_{k,\Omega} = |u|_{k,\Omega} = |u|_{H^k(\Omega)}$ if no subscript index

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is given then we assume an ordinary L_2 norm, $\|.\|_{0,\Omega}$, and if no subscript index is given then we shall assume it is the whole of Ω. To define weak form of (2.1), we need two classes of functions: the trial functions H_E^1 and the test solutions $H_{E_0}^1$:

$H_E^1 = \{u \in H^1(\Omega) : u = g \text{ on } \Gamma_D\}$

$H_{E_0}^1 = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}$

and the standard variational formulation of problem (2.1) is given by: find $u \in H_E^1$ such that

$$B(u,v) = F(v) \quad \forall v \in H_{E_0}^1 \qquad (2.2)$$

where

$B(u,v) = \epsilon(\nabla u, \nabla v) + (\mathbf{b}.\nabla u, v)$ and $F(v) = (f, v)$

Let $\mathcal{T}_h = \{K\}$ be a decomposition of Ω into triangles or quadrilaterals.

We need to make the following geometrical assumptions on the family of triangulations $\mathcal{T}_h$

1. **Admissibility**: whenever $K_1 and\, K_2$ belongs to $\mathcal{T}_h$, $K_1 \cap K_2$ is either empty, or reduced to a common vertex, or to a common edge
2. h_k = the diameter of K = the longest side of $K \in \mathcal{T}_h$
3. ρ_k= the supremum of the diameter of the balls inscribed in $K \in \mathcal{T}_h$
4. **Shape regularity**: the ratio of h_k to ρ_k is uniformly bounded i.e.,
   $\frac{h_k}{\rho_k} \le \beta_k \ \forall K \in \mathcal{T}_h$

which means for any $h > 0$ and for any $K \in \mathcal{T}_h$ there exists a constant $\beta_0 > 0$ such that $\beta_k \ge \beta_0$, where β_k denotes the smallest angle in any $K \in \mathcal{T}_h$. We define the finite element spaces

$V_h = \{v \in H^1(\Omega) : v\big|_K \in P^1(K) \ \forall K \in \mathcal{T}_h\}$

for triangular elements and

$V_h = \{v \in H^1(\Omega) : v\big|_K \in Q^1(K) \ \forall K \in \mathcal{T}_h\}$

for rectangular elements, where $P^1(K)$ is the space of polynomials of degree not greater than 1, and $Q^1(K)$ is the space of polynomials of complete degree 1 on K. In the case of convection – dominated problem, the standard Galerkin approximation of (2.2) may produce unphysical behavior, oscillation, if the mesh is too coarse in critical regions. To circumvent these difficulties, stability of the discretization has to be increased by introducing artificial diffusion along streamlines. The Streamline – Diffusion Finite Element Method (SDFEM) [18] or [14], [15] stabilizes a convection – dominated problem by adding weighted residuals to the standard Galerkin finite element method for hyperbolic equations which combines good stability with high order accuracy, convergence results are available (see [22]). The SDFEM yields the following discrete problem obtained: Find $u_h \in V_h$ such that

$$B(u_h, v_h) = F(v_h) \quad \forall\, v_h \in V_h, v_h = 0 \ on\ \Gamma_D \qquad (2.3)$$

where

$B_{SD}(u_h, v_h) =$
$\epsilon(\nabla u_h, \nabla v_h) + (\boldsymbol{b}.\nabla u_h, v_h) + \sum_{K\in\mathcal{T}_h} \delta_K(\boldsymbol{b}.\nabla u_h, \boldsymbol{b}.\nabla v_h)_K$

and $F_{SD}(v_h) = (f, v_h) + \sum_{K\in\mathcal{T}_h} \delta_K(f, \boldsymbol{b}.\nabla v_h)_K$.

In (2.3), a constant δ_K must be chosen for every element *K.* Let the mesh Peclet number be defined by , $\mathrm{Pe_k} = \|b\|_{\infty,K} h_k/\epsilon$ where $\|.\|_{\infty,K}$ denotes the norm in $(L_\infty(K))^2$. From the analysis of the SDFEM, the following choice of δ_K are optimal; Elman et al. [5]:

$$\delta_k = \begin{cases} \frac{h_k}{2}\left(1 - \frac{1}{\mathrm{Pe_k}}\right) & \text{for } \mathrm{Pe_k} > 1, \\ 0 & \text{for } \mathrm{Pe_k} \le 1, \end{cases} \qquad (2.4)$$

where h_k is a measure of the element length in the direction of the wind. For other parameter choice, see [16], [17], [25], [12].

3 NEUMANN – TYPE A POSTERIORI ERROR ESTIMATION

In this topic, we introduce the analysis of error estimator proposed by Kay and Silvester [24] which is an extension of the work of Verfurth [28]. In their work, they modify the well-known Bank and Weiser estimator [5] and using the idea of Ainsworth & Oden [1], they solve a local (element) Poisson problem over a suitably chosen (higher order) approximation space with data from interior residuals and flux jumps along element edges. We now introduce some definitions and notations that will be needed for the error estimates. We denote by $\mathcal{E}(K)$ the set of edges of element $K \in \mathcal{T}_h$, by $\mathcal{E}_h = \cup_{K\in\mathcal{T}_h} \mathcal{E}(K)$ the set of all element edges and the subsets relating to internal, Dirichlet and Neumann edges respectively as $\mathcal{E}_{h,\Omega} = \{E \in \mathcal{E}_h : E\ C\ \Omega\}$, $\mathcal{E}_{h,D} = \{E \in \mathcal{E}_h : E\ C\ \partial\Omega_D\}$ and $\mathcal{E}_{h,N} = \{E \in \mathcal{E}_h : E\ C\ \partial\Omega_N\}$ so that $\mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,D} \cup \mathcal{E}_{h,N}$. We denote $\mathcal{N}_K$ the set of vertices of $K \in \mathcal{T}_h$ and by $\mathcal{N}_h = \cup_{K\in\mathcal{T}_h} \mathcal{N}_K$ the set of all element vertices (that do not lie on the Dirichlet boundary Γ_D). Let $\mathcal{N}_E$ be the set of vertices of $E \in \mathcal{E}_h$, and for $K \in \mathcal{T}_h$, $E \in \mathcal{E}_h$ and $\mathrm{x} \in \mathcal{N}_h$ we define the local 'patches' of elements as

$\omega_K = \cup_{\mathcal{E}(K)\,\cap\,\mathcal{E}(K')\neq\emptyset} K'$, $\qquad \omega_E = \cup_{E\in\mathcal{E}(K')} K'$,

$\widetilde{\omega}_K = \cup_{\mathcal{N}_K\,\cap\,\mathcal{N}_{K'}\neq\emptyset} K'$, $\qquad \widetilde{\omega}_E = \cup_{\mathcal{N}_E\,\cap\,\mathcal{N}_{K'}\neq\emptyset} K'$.

For the lowest order P^1 or Q^1 approximations over a triangular or rectangular element subdivision, $\Delta u_h\big|_K = 0$, so that the interior residual of element K is given by

$$R_K = (f - \mathbf{b}.\nabla u_h)\big|_K \qquad (3.1)$$

And the internal residual is approximated by

$$R_K{}^0 = \mathcal{P}_K{}^0(R_K) \qquad (3.2)$$

where $\mathcal{P}_K{}^0$ is the $L_2(K)$ – projection onto $P^0(K)$.

For any edge E of an element $K \in \mathcal{T}_h$, we define the flux jump as

$$R_E = \begin{cases} \frac{1}{2}\left[\!\left[\frac{\partial u_h}{\partial n_E}\right]\!\right]_E & if\ E \in \mathcal{E}_{h,\Omega} \\ -\nabla \mathrm{u_h}.\vec{\mathrm{n}}_{\mathrm{E,K}} & \text{if } E \in \mathcal{E}_{h,N} \\ 0 & if\ E \in \mathcal{E}_{h,D} \end{cases} \qquad (3.3)$$

where $\frac{\partial u_h}{\partial n_E}$ is a constant function on the inter-element edge E and $\left[\!\left[\frac{\partial u_h}{\partial n_E}\right]\!\right]$ measures the jump of $\frac{\partial u_h}{\partial n_E}$ across E, that is, for $E \in \mathcal{E}(K) \cap \mathcal{E}(S)$, $K, S \in \mathcal{T}_h$ and defining $n_{E,K}$ and $n_{E,S}$ to be the outward normals with respect to the edge E from element K and S respectively, we have

$$\left[\!\left[\frac{\partial u_h}{\partial n_E}\right]\!\right]_E = \left(\nabla v\big|_K - \nabla v\big|_S\right).\ n_{E,K} = \left(\nabla v\big|_S - \nabla v\big|_S\right).\ n_{E,S} \qquad (3.4)$$

The approximation space is denoted by

$$\boldsymbol{Q_K} = \mathcal{Q}_{\boldsymbol{K}} \oplus \boldsymbol{B_K} \qquad (3.5)$$

consisting of edge and interior bubble functions respectively:

$$\mathcal{Q}_{\boldsymbol{K}} = span\{\psi_E : K \to \mathbb{R}\ \big|\ 0 \le \psi_E \le 1, E \in \mathcal{E}(K) \cap (\mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,N})\} \qquad (3.6)$$

where each member of the space is a quadratic (or biquadratic) edge bubble function ψ_E that is nonzero on edge E of element K, but non zero valued on all other

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edges of K. B_K is the space spanned by interior cubic (or biquadratic) bubbles ϕ_K i.e.,

$$B_K = \{\phi_K : K \to \mathbb{R} \mid 0 \le \phi_K \le 1,\ \phi_K = 0 \ on\ \partial K\} \qquad (3.7)$$

where each function is associated with an element K, and is zero on all edges of K, nonzero on the interior of K, and $\phi_K = 1$ at the centeroid of K. The upshot is that the local problems are always well posed and that for each triangular (or rectangular) element a 4x4 (5x5) system of equations must be solved to compute e_K. For an element $K \in \mathcal{T}_h$, the local error estimate is the energy norm of e_K given by

$$\eta_K = \|\nabla e_K\|_K \qquad (3.8)$$

where $e_K = (u - u_h)\big|_K \in Q_K$ satisfies

$$\epsilon(\nabla e_K, \nabla v)_K = (R_K^0, v)_K - \frac{1}{2}\epsilon \sum_{E\in\mathcal{E}(K)} \langle R_E, v\rangle_E \quad \forall\, v \in Q_K \qquad (3.9)$$

In the following, we make frequent use of the short-hand notation $\|f\|_S$ to denote L_2- norm of a function $L_2(S)$. The Kay and Silvester's upper estimation is the following.

Theorem 1. If the variational problem (2.2) solved with a grid of bilinear rectangular elements, and if the rectangle aspect ratio condition is satisfied with β_Ω, then, the estimator η_K computed via (3.9) safisfies the upper bound property

$$\|\nabla e_h\| \le C(\beta_\Omega)\left(\sum_{K\in\mathcal{T}_h} {\eta_K}^2 + \sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{\epsilon}\|\boldsymbol{b}\|_K\right)^2 \|R_K - R_K^0\|_K^2\right)^{1/2} \qquad (3.10)$$

where C is independent of ϵ and h and h_K is the length of the longest edge of element K.

Proof. Using the assumptions (A.2) - (A.4) we have

$$B_{SD}(e_h, e_h) \ge \epsilon\|\nabla e_h\|_\Omega^2 \qquad (3.11)$$

Given $e_h = u - u_h \in H_{E_0}^1$ there exist a quasi – interpolant $e_h^* \in V_h \cap H_{E_0}^1$ (see [6]) such that

$$\|e_h - e_h^*\|_K \le C_1(\beta_{\tilde{\omega}_K}) h_k \|\nabla e_h\|_{\tilde{\omega}_K} \quad \forall\, K \in \mathcal{T}_h \qquad (3.12)$$

$$\|e_h - e_h^*\|_E \le C_2(\beta_{\tilde{\omega}_K}) {h_E}^{1/2}\|\nabla e_h\|_{\tilde{\omega}_K} \quad \forall\, E \in \mathcal{E}_h \qquad (3.13)$$

Using the Galerkin orthogonality property, the bilinear form $B_{SD}(e_h, e_h)$ is written as

$$B_{SD}(e_h, e_h) = B_{SD}(e_h, e_h - e_h^*) + B_{SD}(e_h, e_h^*)$$
$$= B_{SD}(e_h, e_h - e_h^*) - \sum_{K\in\mathcal{T}_h}\delta_K(f - \boldsymbol{b}.\nabla u_h, \boldsymbol{b}.\nabla e_h^*)_K$$
$$= B_{SD}(u, e_h - e_h^*) - B_{SD}(u_h, e_h - e_h^*) - \sum_{K\in\mathcal{T}_h}\delta_K(f - \boldsymbol{b}.\nabla u_h, \boldsymbol{b}.\nabla e_h^*)_K$$
$$= (f, e_h - e_h^*) - B_{SD}(u_h, e_h - e_h^*) - \sum_{K\in\mathcal{T}_h}\delta_K(f - \boldsymbol{b}.\nabla u_h, \boldsymbol{b}.\nabla e_h^*)_K \qquad (3.14)$$

Hence, using (3.11) in (3.14) and integrating by parts elementwise with $e_h = e_h^*$ on Γ_D gives

$$\epsilon\|\nabla e_h\|_\Omega^2 \le \sum_{K\in\mathcal{T}_h}\Big[(f - \boldsymbol{b}.\nabla u_h, e_h - e_h^*)_K - \delta_K(f - \boldsymbol{b}.\nabla u_h, \boldsymbol{b}.\nabla e_h^*)_K + \frac{\epsilon}{2}\sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Omega}}\left\langle\left[\!\left[\frac{\partial u_h}{\partial n_E}\right]\!\right]_E, e_h - e_h^*\right\rangle_E - \epsilon\sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,N}}\left\langle\frac{\partial u_h}{\partial n_E}, e_h - e_h^*\right\rangle_E\Big]$$
$$= \sum_{K\in\mathcal{T}_h}\Big[(R_K, e_h - e_h^*)_K - \delta_K(R_K, \boldsymbol{b}.\nabla e_h^*)_K + \frac{\epsilon}{2}\sum_{E\in\mathcal{E}_{h,\Omega}\cup\mathcal{E}_{h,N}}\langle R_E, e_h - e_h^*\rangle_E\Big] \qquad (3.15)$$

From coercivity estimate, interpolation estimates (3.12) & (3.13) and the Cauchy-Schwarz inequality, it can be shown

$$\epsilon\|\nabla e_h\|_\Omega^2 \le C(\beta_\Omega)\sum_{K\in\mathcal{T}_h}\Big[h_K\Big(\|R_K^0\|_K + \|R_K - R_K^0\|_K + \frac{\epsilon}{2}\sum_{E\in\mathcal{E}_{h,\Omega}\cup\mathcal{E}_{h,N}} h_K^{1/2}\|R_E\|_E\Big)\|\nabla e_h\|_{\tilde{\omega}_K}\Big]$$
$$\le C(\beta_\Omega)\|\nabla e_h\|_\Omega\left\{\sum_{K\in\mathcal{T}_h} h_K^2\|R_K^0\|_K^2 + h_K^2\|R_K - R_K^0\|_K^2 + \left(\frac{\epsilon}{2}\right)^2\sum_{E\in\mathcal{E}_{h,\Omega}\cup\mathcal{E}_{h,N}} h_E\|R_E\|_E^2\right\} \qquad (3.16)$$

Now, it remains to bound $\|R_K^0\|_K$ and $\|R_E\|_E$ in terms of η_K. Recall that R_K^0 is constant and for $K \in \mathcal{T}_h$, consider a function $\omega_K = R_K^0\phi_K$ where $\phi_K \in B_K$. So ω_K an interior bubble function with maximum value $\|R_K^0\|_K$ at the center of the element. As $0 \le \phi_K \le 1$, one can write

$$\|R_K^0\|_K^2 \equiv (R_K^0, R_K^0)_K = C_1(R_K^0, \omega_K)_K \qquad (3.17)$$

Using (3.9) and the Cauchy-Schwarz inequality, imply that

$$\|R_K^0\|_K^2 \le C_1\epsilon(\nabla e_K, \nabla\omega_K)_K \le C_2\|\nabla e_K\|_K\|\nabla\omega_K\|_K \qquad (3.18)$$

where $C_2 = C_1\epsilon$.

Now using the inverse estimate (see [10, Lemma 1.26])

$$\|\nabla\omega_K\|_K \le C_3 h^{-1}\|\omega_K\|_K \qquad (3.19)$$

and by definition $\|\omega_K\|_K \le \|R_K^0\|_K$, we get

$$\|R_K^0\|_K \le C_4 h^{-1}\|\nabla e_K\|_K \qquad (3.20)$$

where $C_4 = C_2C_3$.

Now we deal with the edge residual term R_E. For $E \in \mathcal{E}(K)$, consider $\omega_E = R_E\psi_E$ where $\psi_E \in \mathbb{Q}_K \subset Q_K$. This is the bubble function that is only nonzero on edge E. Note that we can write each norms of ω_E as norms over the edges as follows (see [23, Lemma 3.2])

$$\|\omega_E\|_{K'} \le {h^{1/2}}_E\|\omega_E\|_E$$
$$\|\nabla\omega_E\|_{K'} \le {h^{-1/2}}_E\|\omega_E\|_E$$

and since $\|\omega_E\|_E < \|R_E\|_E$ we can have

$$\|\omega_E\|_{K'} \le h_E^{1/2}\|R_E\|_E \qquad (3.21)$$

$$\|\nabla\omega_E\|_{K'} \le h_E^{-1/2}\|R_E\|_E \qquad (3.22)$$

Note

$$\epsilon\|R_E\|_E^2 = \epsilon\int_E (R_E)^2\, ds \le C_5\epsilon\int_E R_E(R_E\psi_E)ds \le C_5\epsilon\langle R_E, \omega_E\rangle \qquad (3.23)$$

Using (3.9), (3.21) and (3.22), we have

$$\epsilon\|R_E\|_E^2 \le C_5\sum_{K'\subset\omega_E}\left[\epsilon\|\nabla e_{K'}\|_{K'}\|\nabla\omega_E\|_{K'} + \left\|R_{K'}^0\right\|_{K'}\|\omega_E\|_{K'}\right]$$
$$\le C_5(\beta_{\omega_E})\|R_E\|_E\sum_{K'\subset\omega_E}\left[\epsilon h_E^{-1/2}\|\nabla e_{K'}\|_{K'} + h_{K'}^{1/2}\left\|R_{K'}^0\right\|_{K'}\right] \qquad (3.24)$$

By plugging (3.20) and (3.24) into (3.16), the global upper bound (3.10) holds.

Remark 1. If f and $\boldsymbol{b}$ are both piecewise constant functions then the consistency error term $\|R_K - R_K^0\|_K$ is identically zero. Otherwise, if f and $\boldsymbol{b}$ are smooth, this term represents a high-order perturbation. In any case the estimator η_K is reliable in the sense that the upper bound (3.10) is independent of h and ϵ. Establishing that the estimated error η_K gives a lower bound on the local error is not possible. The difficulty is generic for any local error estimator: the local error is overestimated within exponential boundary layers wherever such layers are not solved by the mesh. Thus, in contrast to the nice lower bound that holds when solving Poisson's equation, see [10, Proposition 1.28], the tightest lower bound that can be established for the convection-diffusion equation is the following.

Theorem 2. If the variational problem (2.2) with $\|\boldsymbol{b}\|_\infty = 1$ is solved via either the Galerkin formulation or the SD formulation (2.3), using a grid of bilinear rectangular elements, and if the rectangle aspect ratio condition is satisfied, then the estimator η_K computed via (3.9) is a local lower bound for $e_h = u - u_h$ in the sense that

$$\eta_K \leq C(\beta_{\omega_K})(\|\nabla e_h\|_{\omega_K} + \sum_{K \subset \omega_K} \frac{h_k}{\epsilon}\|\mathbf{b}.\nabla e_h\|_K + \sum_{K \subset \omega_K} \frac{h_k}{\epsilon}\|R_K - R_K^0\|_K) \quad (3.25)$$

where C is independent of ϵ, and ω_K represents the patch of five elements that have at least one boundary edge E from the set $\mathcal{E}(K)$.

Proof. To begin note that as $e_k \in Q_K$ we can put $v = e_k$ in (3.9) to get

$$\epsilon\|\nabla e_k\|_K^2 = (R_K^0, e_k)_K - \frac{1}{2}\epsilon\sum_{E\in\mathcal{E}(K)}\langle R_E, e_k\rangle_E$$
$$\leq \|R_K^0\|_K\|e_k\|_K + \frac{\epsilon}{2}\sum_{E\in\mathcal{E}(K)}\|R_E\|_E\|\nabla e_k\|_K \quad (3.26)$$

Note that with Poincare – Fridrichs inequality and a scaling argument we can write

$$\|e_k\|_K \leq C(\beta_K)h_K\|\nabla e_k\|_K, \quad (3.27)$$
$$\|e_k\|_E \leq C(\beta_K)h_E^{1/2}\|\nabla e_k\|_K \quad (3.28)$$

Substituting these into (3.26) gives

$$\epsilon\|\nabla e_k\|_K \leq C(\beta_K)\left(h_K\|R_K^0\|_K + \frac{\epsilon}{2}\sum_{E\in\mathcal{E}(K)} h_E^{1/2}\|R_E\|_E\right) \quad (3.29)$$

Now we only need to bound $h_K\|R_K^0\|_K$ and $h_E^{1/2}\|R_E\|_E$ in terms of ∇e_k. For $K \in \mathcal{T}_h$, consider $\omega_K = R_K^0\psi_K$ as above. Then

$$C\|R_K^0\|_K^2 \leq (R_K^0, \omega_K)_K = (R_K^0 - R_K, \omega_K)_K + B_{SD}(u - u_h, \omega_K)$$
$$= (R_K^0 - R_K, \omega_K)_K + \epsilon(\nabla e_h, \nabla\omega_K)_K + (\mathbf{b}.\nabla e_h, \omega_K)_K$$
$$\leq \|R_K^0 - R_K\|_K\|\omega_K\|_K + \epsilon\|\nabla e_h\|_K\|\nabla\omega_K\|_K + \|\mathbf{b}.\nabla e_h\|_K\|\omega_K\|_K \quad (3.30)$$

Hence, using $\|\nabla\omega_K\|_K \leq \|R_K^0\|_K$ and (3.11) we get

$$h_K\|R_K^0\|_K \leq C(h_K\|R_K^0 - R_K\|_K + \epsilon\|\nabla e_h\|_K + h_K\|\mathbf{b}.\nabla e_h\|_K) \quad (3.31)$$

Similarly, for $E \in \mathcal{E}(K) \cap \mathcal{E}_{h,\Omega}$ and by using ω_E, it can be shown

$$C\epsilon\|R_E\|_E^2 \leq \epsilon\langle R_E, \omega_E\rangle_E = \sum_{K'\subset\omega_E}\epsilon(\nabla \mathrm{u_h}, \nabla\omega_E)_{K'},$$ by the definition of R_E and the Green formula

$$= \sum_{K'\subset\omega_E} -\epsilon(\nabla e_h, \nabla\omega_E)_{K'} + \epsilon(\nabla \mathrm{u}, \nabla\omega_E)_{K'}$$
$$= \sum_{K'\subset\omega_E}[\left(R_{K'} - R_{K'}^0 + R_{K'}^0 - \mathbf{b}.\nabla e_h, \omega_E\right)_{K'} - \epsilon(\nabla e_h, \nabla\omega_E)_{K'}]$$

Thus,

$$\epsilon\|R_E\|_E^2 \leq C\|R_E\|_E h_E^{-1/2}\sum_{K'\subset\omega_E}\left[h_{K'}\left\|R_{K'}^0 - R_{K'}\right\|_{K'} + \epsilon\|\nabla e_h\|_{K'} + h_{K'}\|\mathbf{b}.\nabla e_h\|_{K'} + h_{K'}\left\|R_{K'}^0\right\|_{K'}\right] \quad (3.32)$$

Therefore, by (3.31), we have

$$\epsilon h_E^{1/2}\|R_E\|_E \leq C\sum_{K'\subset\omega_E}\left[h_{K'}\left\|R_{K'}^0 - R_{K'}\right\|_{K'} + \epsilon\|\nabla e_h\|_{K'} + h_{K'}\|\mathbf{b}.\nabla e_h\|_{K'}\right] \quad (3.33)$$

By plugging (3.31) and (3.33) into (3.29) the local lower bound (3.25) holds.

Remark 2. The same estimate is obtained if local convection-diffusion problems are solved in place of (3.9), see [29]. The restriction that $\|\boldsymbol{b}\|_\infty = 1$ can be removed; it is only included to simplify the proof. From the bound (3.10) and (3.25), the structure of the "optimality gap" term $\sum_{K\subset\omega_K}\frac{h_k}{\epsilon}\|\mathbf{b}.\nabla e_h\|_K$, leads to the expectation that η_K will be an overestimate for $\|\nabla e\|_K$ in any element K where first, the element Peclet number, $\frac{h_k}{2\epsilon}$, is significantly bigger than unity, and second, the derivative of the error in the streamline direction, namely $\|\mathbf{b}.\nabla e\|_K$, is commensurate with the derivative of the error in the cross wind direction, $\|\mathbf{b}^\perp.\nabla e\|_K$. Such a deterioration in performance is realized in the case of problem given in section (4) next, if ϵ is decreased while keeping the grid fixed.

4 NUMERICAL RESULTS: EFFECTIVITY INDICES

In this section, we present an example to compare the solution qualities from the GK method and the SDFEM and we compute global effectivity indices. In order to see how the effectivity indices change in terms of the diffusive parameter ϵ, mesh size h, and element Peclet number $h_k/2\epsilon$, the problem is solved over uniform meshes with mesh size $h = \frac{1}{8}$ and $\frac{1}{16}$ for $\epsilon = \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}$ and $\frac{1}{4096}$ with $Pe_K = h_k/2\epsilon = 1, 4, 16, 64$ and 256.

Problem 1. Consider the function

$$u(x,y) = x\left(\frac{1-e^{y-1/\epsilon}}{1-e^{-2/\epsilon}}\right) \quad (4.1)$$

satisfies equation (2.1) with $\boldsymbol{b} = (0,1)$ and $f = 0$. Dirichlet conditions on the boundary $\partial\Omega$ are determined by (4.1) and satisfy

$$u(x,-1) = x, \quad u(x,1) = 0, \quad u(-1,y) \approx -1, \quad u(1,y) \approx 1$$

where the latter two approximations hold except near $y = 1$. Clearly, exponential layer near $y = 1$ is expected, the layer is determined by $e^{1-y/\epsilon}$ and has width proportional to ϵ, see Eckhaus (1979) and Roos et al. ((1996), section III.1.3). We solve (2.3) using uniform NxN grids of square elements i.e., Q^1 approximation over a domain $\Omega = [0,1]x\,[0,1]$. To illustrate the qualities of SDFEM and GK method for $Pe_K > 1$, we consider the case N = 8, 16, 32 & 64 for $\epsilon = 1/1024$ and plot isolines of the computed solution and the estimated error in Fig. 2, Fig. 3, Fig 4, Fig. 5 & Fig. 6. Clearly, Fig. 2 shows GK solution suffer with serious oscillation on whole domain but Fig. 3, Fig. 4, Fig. 5 & Fig. 6 show SD solutions maintain good solution quality with small oscillation in the layer region as h increases. However, the solution still contains oscillations.

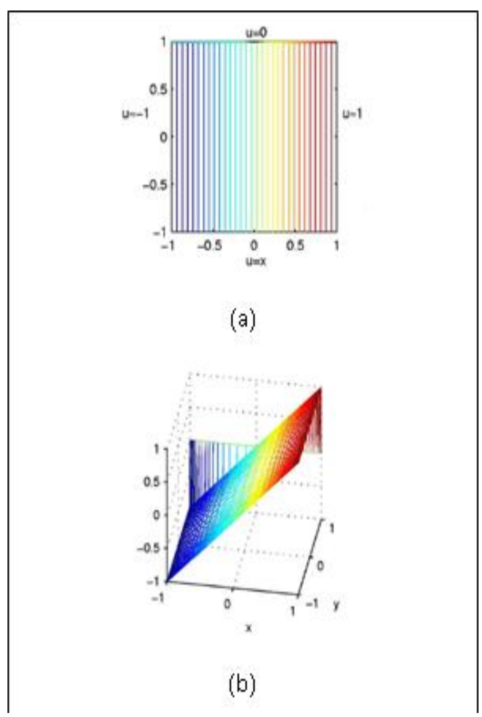

***Fig. 1. (a)** Contour plot and (b) three-dimensional surface plot (bottom) of an accurate FE solution, for $\epsilon = 1/200$.*

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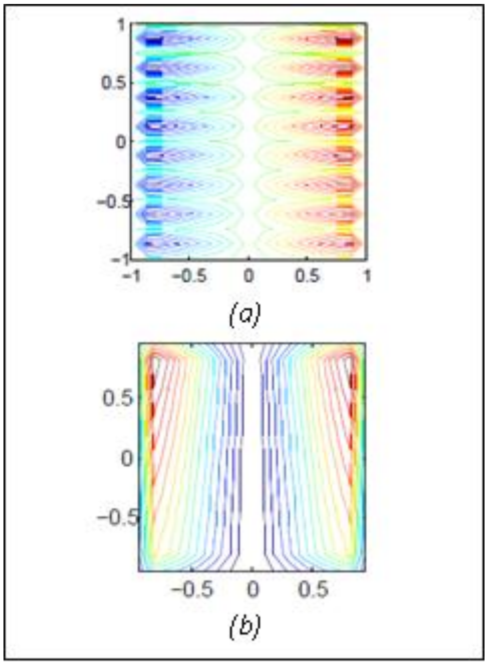

Fig. 2. (a) *GK solution and (b) estimated error: 16 x 16 uniform grid*

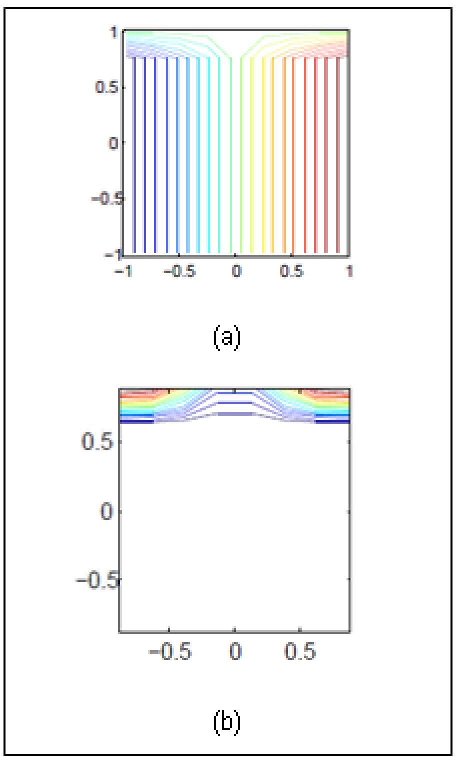

***Fig. 3**. (a) SD solution and (b) estimated error: 8 x 8 uniform grid*

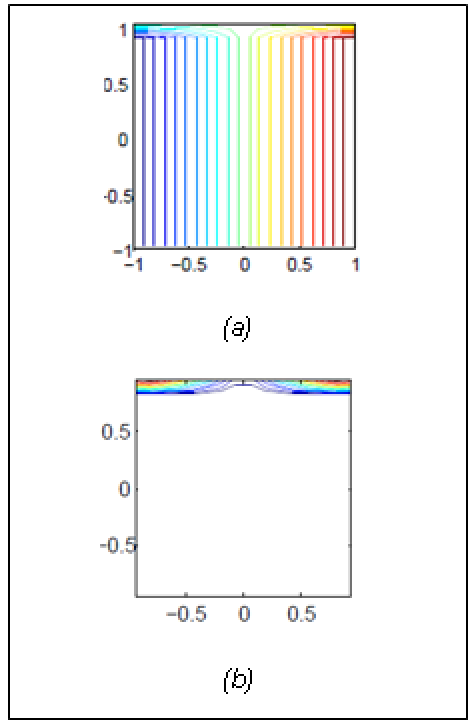

***Fig. 4. (a**) SD solution and (b) estimated error: 16 x 16 uniform grid*

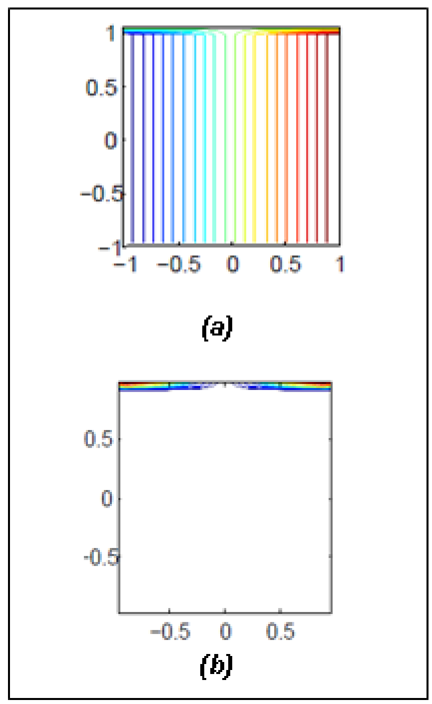

Fig. 5. (a) *SD solution and (b) estimated error: 32 x 32 uniform grid.*

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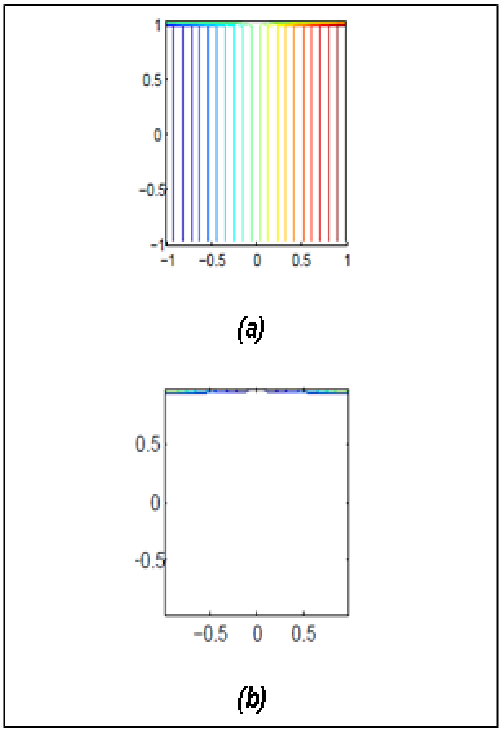

***Fig. 6. (a)** SD solution and (b) estimated error: 64 x 64 uniform grid.*

Then, an approximation to the exact Galerkin error $E_h = \|\nabla e_h\|_K$ and the corresponding SD error $E_h^* = \|\nabla e_h^*\|_K$ on each element can be computed as using Gaussian quadrature, with the MATLAB quadrature function **dblquad.m.** As an example, estimated global errors

TABLE 1

EXACT ERRORS (E_h), ESTIMATED ERRORS (η) AND EFFECTIVITY INDICES (X_η) FOR ELEMENT PECLET NUMBER (Pe_K) SOLVED WITH GALERKIN APPROXIMATION USING 8 X 8 UNIFORM GRID

ϵ	E_h	Pe_K	η	X_η
$1/16$	1.850	1	3.026	3.220
$1/64$	5.616	4	3.217 x 10^1	5.730
$1/256$	1.652 x 10^1	16	4.388 x 10^2	2.65 x 10^1
$1/1024$	5.877 x 10^1	64	6.778 x 10^3	1.15 x 10^2
$1/4096$	2.274 x 10^2	256	1.075 x 10^5	4.73 x 10^2

$\eta = \left(\sum_{K\in\mathcal{T}_h} \eta_K^2\right)^{1/2}$ and corresponding effectivity indices $X_\eta = \eta/E_h$ and $X_\eta^* = \eta^*/E_h^*$, for GK and SD methods are presented in the tables below. From a comparison given in Table 1, Table 2, Table 3 & Table 4, while the SD errors E_h^* are smaller than the GK errors E_h, we observe that the slow reduction in the global errors as the grid is successively refined. From Table 1 and Table 3, we can see that the effectivity indices deteriorates in a rate $O(Pe_K)$ as $h \gg \epsilon$. Suggesting that the bound (3.25) is tight in this instance. Further, the numerical data in Table 2 and Table 4 also show that the global effectivity indices deteriorates in a rate of $O(\sqrt{Pe_K})$ as mentioned in Kay and Silvester (2001) and Elman et al. (2005). Hence, we note that the local error is overestimated within exponential boundary layer. In considering the results in Table 1, Table 2, Table 3 & Table 4, we note that the improvement in performance of SD over Galergin might be anticipated from (3.25) – in particular the better approximation of the streamline derivative suggests that the gap term will be smaller for SD in the limit as $Pe_K \to \infty$.

TABLE 2

EXACT ERRORS E_h^*, ESTIMATED ERRORS (η^*) AND EFFECTIVITY INDICES (X_η^*) FOR ELEMENT PECLET NUMBER (Pe_K) SOLVED WITH SD APPROXIMATION USING 8 X 8 UNIFORM GRID.

ϵ	E_h^*	η^*	X_η^*	Pe_K
$1/16$	1.675	2.114	1.260	1
$1/64$	4.335	8.852	2.040	4
$1/256$	9.101	3.613 x 10^1	3.970	16
$1/1024$	1.841x10^1	1.454 x 10^2	7.900	64

TABLE 3

EXACT ERRORS (E_h), ESTIMATED ERRORS (η) AND EFFECTIVITY INDICES (X_η) FOR ELEMENT PECLET NUMBER (Pe_K) SOLVED WITH GALERKIN APPROXIMATION USING 16 X 16 UNIFORM GRID

ϵ	E_h	Pe_K	η	X_η
$1/16$	1.85	1	1.515	1.279
$1/64$	4.917	4	1.578 x 10^1	3.210
$1/256$	1.255 x 10^1	16	1.649 x 10^2	1.314 x 10^1
$1/1024$	3.720 x 10^1	64	2.311 x 10^3	6.213 x 10^1
$1/4096$	1.347 x 10^2	256	3.627 x 10^4	2.693 x 10^2

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TABLE 4
EXACT ERRORS E_h^*, ESTIMATED ERRORS (η^*) AND EFFECTIVITY INDICES (X_η^*) FOR ELEMENT PECLET NUMBER (Pe_K) SOLVED WITH SD APPROXIMATION USING 16 X 16 UNIFORM GRID.

ϵ	E_h^*	η^*	X_η^*	Pe_K
$1/16$	1.185	1.515	1.279	1
$1/64$	4.006	6.599	1.647	4
$1/256$	8.948	2.739 x 10^1	3.061	16
$1/1024$	1.833 x 10^1	1.108 x 10^2	6.045	64
$1/4096$	3.688 x 10^1	4.445 x 10^2	1.205 x 10^1	256

5 CONCLUSION

From our theoretical and numerical results, we observe that the estimated error continuously reduces as the grid is successively refined. Therefore, the streamline-diffusion stabilization using N = 32 & 64 leads to the reliable error estimator for all $\epsilon \geq O(10^{-3})$. In the context of the incompressible fluid flow models, exponential boundary layers only arise when downstream boundary conditions are inappropriately specified. In particular, a “hard” Dirichlet boundary condition on an outflow boundary should never be imposed; a zero Neumann condition is invariably more appropriate [13]. To increase accuracy of the solution in the region containing layer, adaptive mesh refinement and mesh movement based on a posteriori error estimation for the convection-diffusion equation are topics of our future work.

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