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Finding Of The Optimum Investment Portfolio Of The Insurance Company With The Use Of Utility Function

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ABSTRACT: In this paper numerical results for the optimization problem of the shares distribution in the investment portfolio of the insurance company are provided , basing on the historical period results, and taking into account the profitability of individual stocks and the yield of the whole portfolio. Optimized portfolio is obtained taking into account minimal risk in the form of the VaR-indicator. The stock market game scheme is provided basing on stock prices in the portfolio. Taking into account the utility function, lines and surfaces of level are built. The strategy minimizing risk and maximizing utility function is received.

Keywords: Utility function, Profitability, Risk, Optimum, Indifference curves, Surface of level.

1 INTRODUCTION

In order to make optimal decisions under uncertainty conditions, taking into account risk and return, we can use the utility theory elements. The utility reflects the subject's degree of satisfaction by certain goods. Using different types of utility functions, we can find estimates of the various economic situations by finding the expected value of the function. J.Neumann and O.Morgenstern [1] showed that the individual decision-maker strives to maximize expected utility. Thus, the utility is a number that corresponds to a possible outcome, and the Neumann-Morgenstern utility function shows the utility of this outcome. Each individual decision-maker has his own utility function, which shows his preference to certain outcomes, depending on its attitude to risk. There is a notion of expected utility of the event. It is equal to the sum of outcomes probabilities multiplied on the utility value of these outcomes. There are three types of people in the economy depending on their attitude to the risk: the risk supporters, opponents and persons neutral to risk [2]. Person neutral to risk is a person who at a given expected result prefers indifference to risk rather than other alternatives. Risk supporters are persons who at a given expected result prefer the alternative associated with risk. The risk opponent is the kind of person who at a given expected result prefers the risk-free alternative to the risky one. The risk opponents have low marginal income utility. As a rule in the economy most people are risk opponents. Neumann-Morgenstern utility function for these types of people is as follows: strictly convex, which every arc of the curve lies below its chord (the risk supporters); strictly concave (the risk opponents); a straight line (neutral to risk). The behavior of most firms in the financial decision-making process can be divided into two types: activity with the adoption of risk-taking and the passive activity with elements of risk. The degree of exposed risk depends not only on the objective economic conditions of the firm existence, but also on the subjective perception of the decision-maker. The utility function must be built taking into account all the objective and subjective conditions that affect consumer preferences. Each preference has its own utility function. Let us consider some of the theorems and definitions which are necessary in the utility function construction.

Definition 1. Let the preference relation $\succeq$ is defined in R_+^n. Any function $u: R_+^n \to R^1$ such that $u(x) \geq u(y)$ if and only if $x \succeq y$, is called a utility function corresponding to this preference relation. If the consumer interests are limited to the set $X \subset R_+^n$, the utility function is defined on this set $u: X \to R^1$. In terms of the utility function indifference relation is given by the equality $u(x) = u(y)$.

Theorem 1. For any preference relation defined and continuous in R_+^n it is possible to construct its continuous utility function $u: R_+^n \to R^1$. It turns out that for any continuous preference relation we can construct a set of the utility functions.

Theorem 2. Let $u: R_+^n \to R^1$ is a utility function representing a preference relation $\succeq$. For any strictly increasing function $f: R^n \to R^1$ a complex function (superposition) $u(x) = f(u(x))$ is a utility function, which also represents a preference relation $\succeq$. The advantage of the utility function concerning other functions which characterize well-being (e.g., the preference relations) is that we can use the differentiation apparatus for the consumer choice analysis. Let the utility function is differentiable

$$\frac{\partial u}{\partial x_i} > 0, i = \overline{1, n} \quad (1)$$

Partial derivative of (1) is called the marginal utility of goods of i type and it determines the utility obtained from the "extra" share of the goods of i type. Inequality (1) can be interpreted as follows: for any set of goods $x \in R_+^n$ increase in the consumption of goods of *i* type at constant level of other goods consumption increases utility. Thus, (1) provides a condition for unsaturatability for a differentiable utility function.

$$H = \begin{Bmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} & .... & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \frac{\partial^2 u}{\partial x_2 \partial x_1} & \frac{\partial^2 u}{\partial x_2^2} & .... & \frac{\partial^2 u}{\partial x_2 \partial x_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \frac{\partial^2 u}{\partial x_n \partial x_2} & .... & \frac{\partial^2 u}{\partial x_n^2} \end{Bmatrix}$$

Suppose that function *u* is twice differentiable and has continuous second partial derivatives. For such function, property of strict concavity is satisfied if the Hessian matrix *H* is negatively definite.

$$\frac{\partial^2 u}{\partial x_i^2} < 0, \quad (i = \overline{1,n}) \qquad (2)$$

Then, in particular, the following conditions are satisfied: $\frac{\partial u}{\partial x_i}$
Inequality (2) says that the marginal utility of goods declines as the product is consumed. Inequalities (1) and (2) reflect the well-known economic law of diminishing marginal utility (the Gossen law). The concept of utility function is inextricably linked with the concept of indifference curves or lines of level.

Definition 2. Indifference curve for a given set of goods $x \in R_+^n$ is the locus of points $y \in R_+^n$ which are in relation of indifference with the set x, i.e. it is a set $y \in R_+^n \mid u(y) = u(x)$. Since all points of this set have the same utility, then the indifference curves are given by equations $u(x) = c$, where $c = const$. Thus, the indifference curve is mathematically represented as a utility function line of level. Therefore, for any utility function there exists an infinite number of indifference curves, and they fill all the space R_+^n, forming the so-called indifference card. Let us exemplify some of the most frequently used utility functions. These functions, as practice proves, under certain conditions, reflect the consumer choice preferences quite objectively.

1. Utility function with the full co-substitutivity:

$$u(x) = \sum_{i=1}^{n} b_i x_i$$

where the coefficient b_i is a numerical evaluation of the utility from the unit of the *i* commodity consumption.

2. Utility function with the full co-complementarity:

$$u(x) = \min\left\{\frac{x_i}{b_i}, i = \overline{1,n}\right\}$$

where b_i is the number of product of *i* type, per unit of utility.

3. Neoclassical utility function (Cobb-Douglas function):

$$u(x) = a\prod_{i=1}^{n} x_i^{b_i}, \sum_{i=1}^{n} b_i = 1$$

where *a* is the factor of utility measurement scale, $0 < b_i < 1$.

4. The utility function of substitute-complementary types:

$$u(x) = \sum_{i=1}^{n} v_i(x)$$

where the functions v_i are found from the system of inequalities

$$x_i \geq \sum_{j=1}^{n} b_j v_j(x), i = \overline{1,n}$$

$$u(x) = \sum_{i=1}^{n} a_i \log(x_i - b_i)$$

5. Logarithmic utility function (Bernoulli function):

$$u(x) = \frac{1}{a}\exp(-w(x))$$

Where $a_i > 0, x_i > b_i \geq 0$.

In 1960 Grayson [3] published results of the study on the restoration of the entrepreneurs utility functions. It was shown that in many cases the investors preferences can be described by the logarithmic utility function.

6.Exponential utility function:

$$a > 0, w(x) = \sum_{i=1}^{n} a_i x_i$$

where.

In the process of various mathematical models construction we can not guarantee the suitability of the known functions for each particular case. In the consumer problem modeling it is required not to choose, but to create utility function for a specific problem. The most common methods for the utility functions construction are the methods of regression analysis, which are applicable in case of a suitable statistical material availability. For the selected type of utility function its parameters are estimated on the basis of these data. The complexity of the method depends on the class of functions (linear, quadratic, power, etc.), with the help of which the utility function is built.

2 Problem Statement

Let's consider the formation and use of the utility function with respect to securities portfolio. This paper does not consider issues related to the nature and type of securities in the portfolio. It is assumed that the portfolio has been created, and it is necessary to find the optimal distribution of its components and to construct utility function. During the portfolio formation we should consider the financial instrument efficiency, taking into account the financial security of investments. The investment portfolio is a collection of securities purchased for income reception and liquidity maintenance [4]. Portfolio management means supporting the balance between liquidity and profitability. Portfolio type is its investment characteristic, based on the ratio of risk and return. In the financial world there are many technologies for risk assessment. Among them there are the following: Value-at-Risk (VaR), beta analysis of the CAPM theory, APT, Short Fall, Capital-at-Risk, Maximum Loss, and several other classical methods. As a risk measure we use the VaR index [5]. The essence of this indicator is that it gives an unambiguous answer to the question that arises at carrying out of financial operations: what is the maximum loss incurred by the investor at risk for a certain period of time with a given probability? Thus, the standard for broker-dealers report on transactions in derivatives, transmitted to the Commission for the Securities and Exchange (USA) is a two-week period and 99% probability. The Bank of International Settlements for bank capital adequacy assessment established the probability of 99% and a period of 10 days. JP Morgan publishes its daily VaR values at 95% confidence level. During the financial instruments portfolio formation questions related to the effective investment of funds arise to the investors. Liquid securities included in the portfolio provide an additional source of income, which is particularly important for the bank management and stockholders. Investments in securities exempt from taxation reduce the bank taxes. An additional source of liquidity are the banks' investments in securities. Purchase of securities with high liquidity partially offsets higher risks of credit portfolio. The advantage of portfolio investment is the ability to choose the type of portfolio.Type of portfolio is its investment characteristic, which is based on the ratio of risk and return. An important feature for the portfolio formation is due to what source of income revenue is received. In this paper it is assumed that only source of income is the market value of the securities included in the portfolio. During the optimal portfolio formation and the utility function construction additional revenues in the form of dividends, interest, etc. are not considered. Investment management process can be divided into five stages:

- investment objectives formulation;
- formation of financial institution investment policy for achieving of the chosen goals;
- choice of the portfolio strategy (active or passive);
- assets selection for inclusion in the portfolio and their optimization;
- portfolio management and evaluation of investments efficiency.

H.Markovitz [6] developed a mathematical model that shows how investors can maximally reduce the portfolio risk at a given rate of return. Using these ideas, we form a portfolio of securities (shares). As a measure of risk we use the VaR indicator, and under the efficiency we understand the exchange stock rate of returns.

Let's consider some problem statements.

Suppose there is some capital with the help of which we can create a portfolio of *n* types of shares, the weight of which depends on the initial capital distribution $x_i \ (i=\overline{1,n})$. The optimal distribution depends on the optimization problem type. Let's consider some of them.

A. Maximum efficiency portfolio.

It is necessary to find the shares x_i of the initial capital allocation which maximize the expected portfolio efficiency E_p, taking into account its individual components efficiencies E_i, provided that the specified portfolio risk value VaR^* is given and the balance condition is satisfied.

$$E_p = \sum_{i=1}^{n} x_i E_i \to \max$$
$$\begin{cases} \sqrt{PVaR^T \Omega PVaR} = VaR^* \\ \sum_{i=1}^{n} x_i = 1; \ x_i \ge 0 \end{cases} \quad (3)$$

where *PVaR* is column vector of individual risk positions; Ω is the correlation matrix.

B. Maximum utility portfolio.

It is necessary to find the shares x_i of the initial capital distribution which minimize the portfolio risk, taking into account maintenance of the required portfolio efficiency level and satisfaction of the balance condition.

$$\sqrt{PVaR^T \Omega PVaR} \to \min$$
$$\begin{cases} \sum_{i=1}^{n} x_i E_i = E^* \\ \sum_{i=1}^{n} x_i = 1; \ x_i \ge 0 \end{cases} \quad (4)$$

Solving the problem (4) for different values of E^* we obtain a set of efficient portfolios.An investor always chooses a portfolio that lies on the efficient frontier. This choice is made by the ratio of risk and return analysis. Thus, we obtain the efficient set of portfolios. Portfolio selection from the effective set depends on the investor's attitude to risk. Investor's preferences considering risk and return can be represented in the form of utility function. Utility function of risk-opponents can be exemplified by a function proposed by M. Rubinstein [7]:

$$U = \psi E - \sigma^2 \quad (5)$$

where ψ is a coefficient that characterizes the individual investor preferences considering risk and return, E is a profitability, σ^2 is risk indicator. In order to choose the most appropriate portfolio from an effective set, investor must draw his indifference curves constructed on the utility function basis, on the same graph with the effective set curve. The optimal portfolio will correspond to the point where the indifference curve is tangent to the effective set, and the utility function reaches its maximum.

3 Numerical Examples

As an example, let's consider a portfolio formed at 23.03.2011 which consists of six companies shares [8]:

ISSN 2347-4289

- Gazprom, common stock (GAZPS);
- Lukoil, common stock (LKOHS);
- Sberbank of Russia, common stock (SBERS);
- Norilsk Nickel, common stock (GMKNS);
- Rosneft, common stock (ROSNS);
- Surgutneftegaz, common stock (SNGSS).

It is necessary to assess the market risk by VaR calculation both for the entire portfolio and for individual companies while investing for a specified number of days. For the analysis we use the historical average prices of our portfolio shares in the Russian Trading System (RTS) with the daily VaR calculation depth being 01.01.2010–23.03.2011. Let's suppose we have initial capital of K = 1000000 conventional units and its distribution is uniform between the issuers. In accordance with this distribution the number of each company shares is calculated. Also we find historical portfolio market value and daily market value changes. Analyzing the given data, we can find average daily price changes and (daily) volatilities of price changes for each company. Using correlation and covariance analysis, we find the correlation matrix of daily price changes (table 1).

Table 1 Correlation matrix of daily price changes

	GAZP	LKOHS	SBERS	GMKNS	ROSNS	SNGSS
GAZP	1,00	0,64	0,62	0,62	0,67	0,50
LKOHS	0,64	1,00	0,54	0,52	0,60	0,57
SBERS	0,62	0,54	1,00	0,59	0,62	0,48
GMKNS	0,62	0,52	0,59	1,00	0,60	0,45
ROSNS	0,67	0,60	0,62	0,60	1,00	0,58
SNGSS	0,50	0,57	0,48	0,45	0,58	1,00

Let's consider the exchange rate of return on the i-th share ($i = \overline{1,n}$), which is calculated by the following formula:

$$E_i = \frac{P_i^1 - P_i^0}{P_i^0} \qquad (6)$$

where P_i^1 is the share value at the end of the period; P_i^0 is the share price at the beginning of the period. Let's consider the optimization problem at 23.03.20011, for the issuers, taking into account the historical period in the form of:

$$\sqrt{PVaR^T \Omega PVaR} \to \min$$

$$\begin{cases} \sum_{i=1}^{n} x_i = 1; \\ x_i \geq 0 \end{cases}$$

Calculation results are presented in Table 2.

Table 2 Initial optimum distribution of a portfolio

	GAZP	LKOHS	SBERS	GMKNS	ROSNS	SNGSS	Portfolio
VaR	7217	3228	0	0	5398	6031	18348
Stock share	0,343	0,140	0	0	0,252	0,263	1

The obtained data allow to allocate the initial capital between the issuers with minimal portfolio risk. Taking into account the market dynamics and the exchange rate of purchase and sale of the shares, we obtain the optimal redestribution of the original portfolio. In accord with this, a new portfolio with a constant initial amount and the same set of shares is formed. Taking into account current stock prices and changes in shares of investment in these stocks, we get a market game strategy in the form of purchase and sale of initial stocks. As a result of this operation, we can find a new portfolio efficiency as follows:

$$E_p = \sum_{i=1}^{n} E_i x_i$$

Through the market game in the form of this strategy, during the period, for example, from 24.03.2011 to 16.06.2011, we obtain statistical data regarding the portfolio formation on the market in the form of minimal risk (VaR-indicator of portfolio returns). In general, the optimal strategy does not guarantee making a profit. Therefore, investors can decide about the purchase and sale of shares in the portfolio, taking into account the maximum financial gain. As an example, the profit for the three-month studied period, may be about 15,000 conventional units , representing 1,5% of the initial capital invested in the portfolio. Providing an analogy with H.Markovitz portfolio theory, for each of the newly formed portfolio we can find the optimal curve "return-risk". As an efficiency criterion we will consider the indicator *E*, which is calculated by the formula:

$$E = \frac{E_p - \overline{E}}{\sigma_E} \qquad (7)$$

where $\overline{E}$ is the average efficiency for the studied period; σ_E is the efficiency's RMS for the studied period. As a risk indicator we will consider an indicator V as follows:

$$V = \frac{VaR - \overline{V}}{\sigma_{VaR}} \qquad (8)$$

where $\overline{V}$ is the average value of the portfolio VaR indicator during the studied period; σ_{VaR} is the RMS of VaR during the

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studied period. Figure 1 shows the distribution of the optimal curve if financial transaction is carried out on 16.06.2011. Auxiliary numerical data have the following values:

$\overline{E}$ =-5,61; σ_E =4,96; $\overline{V}$ =18505,5; σ_{VaR} =264,94

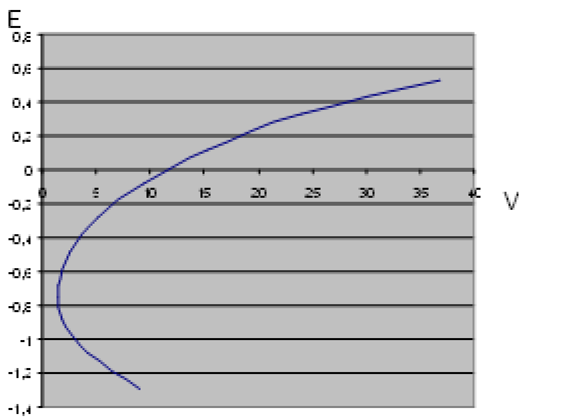

Figure 1: Optimal curve of risk and return.

The curve in Figure 1 can be expressed analytically:

$$E = \begin{cases} -0{,}859 + 0{,}3556Ln(V), (R^2 = 0{,}9952), \ E \geq -0{,}7574 \\ -0{,}6865 - 0{,}2749Ln(V), (R^2 = 0{,}983), \ E < -0{,}7574 \end{cases}$$

Let's construct a utility function $U(E,V)$ for risk-opponents in the form of (5).

$$U = \psi E - V^2 \qquad (9)$$

We take coefficient ψ = 12.48 from the statistical data analysis for the studied period. For (4) in real coordinates "return-risk", we construct the lines of level $U(E,V) = C$, where *C* = *const* (Figure 2) and the surface of level (Figure 3).

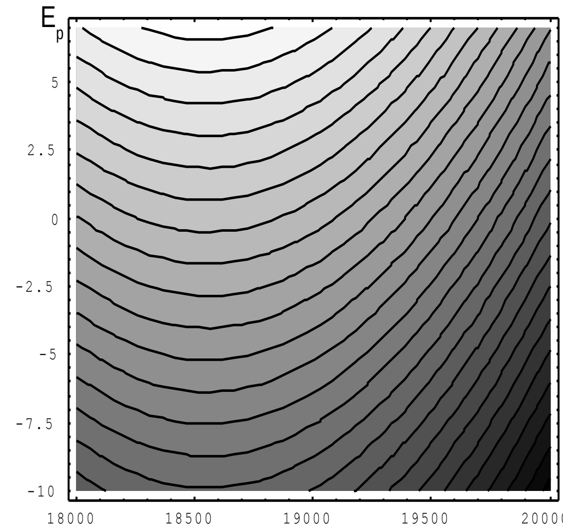

Figure 2: Lines of level

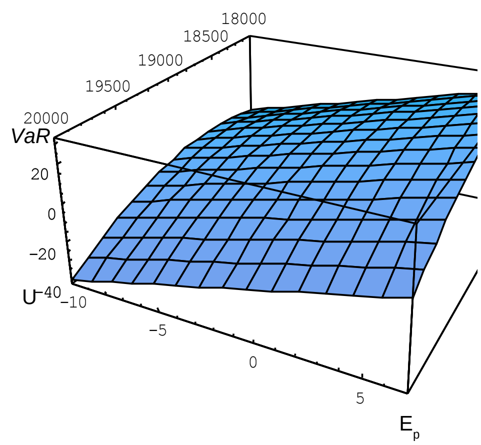

Figure 3 Surface of level

Analyzing the values of the level curves obtained in Figure 2, we can see that the utility function increases with the portfolio efficiency increase. For the utility function $U = \psi E - V^2$ we also construct the corresponding lines of level and surface of level in the system of dimensionless coordinates.

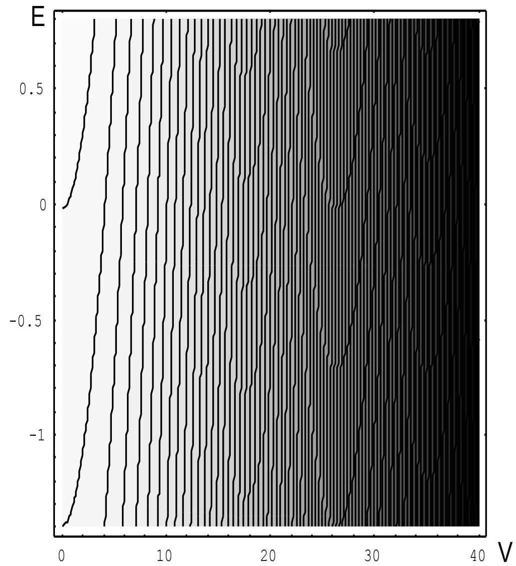

Figure 4: The utility function lines of level

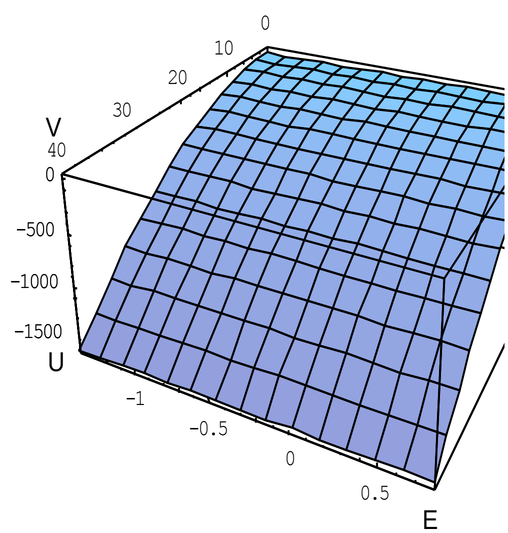

Figure 5 The utility function surface of level

Combining the values obtained in Figure 1 and Figure 4 into the Figure 6, and finding the tangency point of indifference curves to the optimum return-risk curve (point A), we obtain the optimal capital distribution between the issuers taking into account the utility function.

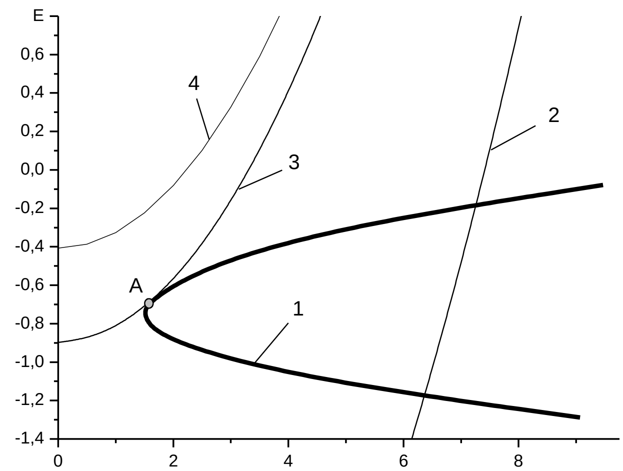

Figure 6 Optimal curve and indifference curves

In Figure 6, curve 1 corresponds to the optimal portfolio, curves 2,3,4 correspond to the utility function levels of line in ascending order. At the tangency point A, we have E_p =-9,18%, VaR = 18899,15 conventional units. The optimal capital allocation between the issuers at the given portfolio efficiency and risk is shown in Table 3:

Table 3: Optimal portfolio allocation at the tangency point A.

	GAZP	LKOHS	SBERS	GMKNS	ROSNS	SNGSS	Portfolio
VaR	4509	3822	0	1213	6920	6076	18899
Stock share	0,200	0,167	0	0,042	0,323	0,265	1

The results obtained in Table 3 allow us to find the optimal distribution of the initial capital between the issuers considering the risk minimization and the utility function maximization.

4 Conclusion

The results of this study allow investors to obtain the optimal strategies in the stock market concerning risk minimization. The presented dependence of risk and return provides a real optimal balance between these factors. Levels of line on the basis of the utility function help determine the optimal funds allocation between the issuers,when the risk minimization and the utility function maximization are achieved.

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