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Approximate Fixed Point Theorem In Generalized Probabilistic 2-Normed Spaces

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ABSTRACT: The conditions imposed in the fixed point theorem in Probabilistic Spaces are too strong. In this paper we steadied Probabilistic Analysis and recall Brouwer's famous fixed point theorem to introduce the existence of approximate fixed point theorem in Generalized Probabilistic 2-Normed Spaces with weaker condition**.**

Keywords: Approximate Fixed point, Probabilistic Spaces, Generalized Probabilistic 2-Normed Spaces, D-boundedness.

1 INTRODUCTION

K. Menger [2] introduced the notion of probabilistic metric spaces. The concept of probabilistic developed by many mathematicians. This field is important as a generalization of deterministic results of linear normed spaces and also in the study of random operator equations. Probabilistic normed spaces were introduced by Serstnev [3]. Then Golet [4] defined Generalized Probabilistic 2-Normed Spaces (GP2N). Some other definitions related to probabilistic normed spaces was studied by Alsina, Schweizer and Sklar [5]. Fixed point theorems are useful in many applied areas. For many practical situations, there is no guarantee that a fixed point exists. In this paper we start from previous obtained definitions and results and then introduce the notion of approximate fixed point and existence of it (approximate fixed point) in Generalized Probabilistic 2-Normed Spaces. Our result extend [1]. Intuitively an approximate fixed point p' of a function f has the property that f(p') is "near to" p in a sense to be specified.

2 PRELIMINARIES

The reference for following section is taken by [6]. Let R^+ denotes the set of real numbers and I denotes the closed unit interval, $R^+ = \{x \in R : x \geq 0\}$, I = [0, 1]. A mapping $F : R^+ \to I$ is called a distribution function if it is non decreasing, left continuous on its domain with inf F = 0 and sup F = 1. The collection of all distribution function denoted by Δ^+ and the subset of those d.f.'s such that F(0) = 0 will be denoted by $D^+ \subseteq \Delta^+$ is defines as

$$D^+ = \{ F \in \Delta^+ : \underset{x\in R^+}{\text{sub}} F(x) = 1\}$$

Let F, G be in D^+ , then we write $F \leq G$ if $F(t) \leq G(t)$ for all $t \in R^+$. If $a \in R^+$, then H_a will be an element of D^+, defined by $H_0(x) = 0$ if $x \leq 0$ and $H_0(x) = 1$ if $x > 0$. A t-norm (triangular norm) T is a two place function $T : I \times I \to I$, which is associative, commutative, non decreasing in each place and such that T(a, 1) = a for all $a \in I$. A triangular function τ is a continuous binary operation on Δ^+ which is associative, commutative, non decreasing and has H_0 as identity. That is, for all $F \in \Delta^+$, we have $\tau (F, H_0) = f$ The typical triangle function is the operation τ_T given by $\tau_T (F,G)(x) = \underset{x\in u+v}{sub} T (F(u), G(v))$ where $F, G \in \Delta^+$

Definition 2.1[1]: A Probabilistic metric space (PM space) is a triple (L, F, τ), where L is a non empty set , F is a mapping $F : L \times L \to \Delta^+$ (probabilistic metric) and τ is a triangle function. Denoting the value of F at the pair (p, q) by F_{pq}, the following conditions are assumed to hold for all $p, q, r \in L$:

(PM1) $F_{pq} = H_0 \Leftrightarrow p = q$
(PM2) $F_{pq} = F_{qp}$
(PM3) $F_{pr} \geq \tau (F_{pq}, F_{qr})$

If $\tau = \tau_T$, then the triple (L, F, τ) is called Menger space.

Definition 2.2[1]: A Probabilistic normed space (PN space) is a triple (L, F, τ), where L is a real linear space (real vector space), F is a mapping $F : L \to \Delta^+$ (probabilistic norm) and τ is a continuous triangle function. Denoting the value of F at $p \in L$ by F_p, the following conditions are assumed to hold for all $p, q \in L$

(PN1) $F_p = H_0 \Leftrightarrow p = \theta$ (null vector in L)
(PN2) $F_p = F_{-p}$
(PN3) $F_{p+q} \geq \tau (F_p, F_q)$
(PN4) $F_p \leq \tau (F_{\alpha p}, F_{(1-\alpha)p})$ for $\alpha \in R - \{0\}$
(PN5) $F_{\alpha p} = F_p\left(\frac{t}{|\alpha|}\right)$ for $t \in R > 0$

Remark: Every PN space is a PM space under F defined by $F_{pq} = F_{p-q}$.

Definition 2.3[4]: A 2-normed space (2-N space) is a pair (L, F), where L is a linear space of a dimension greater than one and F is a real valued mapping on $L \times L$ such that the following conditions be hold for all $p, q, r \in L$

(2-N1) $F_{p,q} = 0$ (iff p and q are linearly dependent)
(2-N2) $F_{p,q} = F_{q,p}$
(2-N3) $F_{\alpha p,q} = |\alpha| (F_{p,q})$ for $\alpha \in R - \{0\}$
(2-N4) $F_{p+q,r} \leq F_{p,r} + F_{q,r}$

Definition 2.4[4]: A Probabilistic 2-normed space (P 2-N space) is a triple (L, F, τ), where L is a linear space of a dimension greater than one and F is a real valued mapping $F: L \times L \to D^+$ (probabilistic 2-norm) and τ is a triangle function such that the following conditions are satisfied for all $p, q, r \in L$

(P2-N1) $F_{p,q} = H_0$ (if p and q are linearly dependent)
(P2-N2) $F_{p,q} \neq H_0$ (if p and q are linearly independent)

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(P2-N3) $F_{p,q} = F_{q,p}$
(P2-N4) $F_{\alpha p,q}(t) = F_{p,q}\left(\frac{t}{|\alpha|}\right)$ for $t \in R > 0$, $\alpha \in R - \{0\}$
(P2-N5) $F_{p+q,r} \geq \tau\,(F_{p,r}, F_{q,r})$

The inequality (P2-N5) can be formulate using a t-norm T
(P2-N6) $F_{p+q,r}(t_1 + t_2) \geq T\,\{F_{p,r}(t_1), F_{q,r}(t_2)\}$ for $t_1, t_2 \in R > 0$ Then (L, F, T) called Random 2-normed space (R 2-N space).

Definition 2.5[4]: Let L and M be two real linear spaces of dimension greater than one, F is a mapping define on the cartesian product L × M such that $F : L \times M \rightarrow D^+$ (generalized probabilistic 2-norm) and τ is a triangle function then a triple (L × M, F, τ) called Generalized probabilistic 2-normed space (GP 2-N space) if the following conditions are satisfied

(GP2-N1) $F_{\alpha p,q}(t) = F_{p,\alpha q}(t) = F_{p,q}\left(\frac{t}{|\alpha|}\right)$ for $t \in R > 0$, $\alpha \in R - \{0\}$, $(p, q) \in L \times M$

(GP2-N2) $F_{p+q,r} \geq \tau\,(F_{p,r}, F_{q,r})$ for $p, q \in L$, $r \in M$
(GP2-N3) $F_{p,q+r} \geq \tau\,(F_{p,q}, F_{p,r})$ for $p \in L$, $q,r \in M$

The inequality (GP2-N2) and (GP2-N3) can be formulate using a t-norm T

(GP2-N4) $F_{p+q,r}(t_1 + t_2) \geq T\,(F_{p,r}(t_1), F_{q,r}(t_2))$ for $t_1, t_2 \in R > 0$, $p, q \in L$, $r \in M$
(GP2-N5) $F_{p,q+r}(t_1 + t_2) \geq T\,(F_{p,q}(t_1), F_{p,r}(t_2))$ for $t_1, t_2 \in R > 0$, $p \in L$, $q,r \in M$

Then (L, F, T) called Generalized random 2-normed space (GR 2-N space). We take the reference of [7] for following sections.

Definition 2.6: The probabilistic diameter $D_{A \times B}$ of any non empty subset $A \times B \subset L \times M$ of GP 2-N space (L × M, F, τ) is define by

$D_{A \times B}(x) = \sup_{t<x} \inf_{(a,b)\in A \times B} F_{a,b}(t)$ and $D_{A \times B}(\infty) = 1$
Then the set A × B called bounded iff $D_{A \times B} \in D^+$, i.e.
$\sup_{x\in(0,+\infty)} \inf_{(a,b)\in A \times B} F_{a,b}(x) = 1$

Theorem 2.1: The probabilistic diameter $D_{A \times B}$ for any non empty subset of GP 2-N space (L × M, F, τ) has the following properties ;

(i). $D_{A \times B} = \begin{cases} \Phi_{A \times B}(x), & \text{if } x \in [\,0,+\infty) \\ 1, & \text{if } x = +\infty \end{cases}$

Where $\Phi_{A \times B}(x) = \inf\{\, F_{a,b}(x) : a \in A, b \in B \,\}$

(ii). $F_{a,b} \geq D_{A \times B}$

(iii). $A \times B \subseteq C \times D \Rightarrow D_{A \times B} \geq D_{C \times D}$

Similarly by [8], we obtain the boundedness in GP 2-N space and state that in any GP2 -N space (L × M, F, τ) for any $A \times B \in L \times M$, D-bound condition is $\lim_{x\rightarrow\infty} D_{A \times B}(x) = 1$

Definition 2.7: For any non empty subset A × B of GP 2-N space (L × M, F, τ) its probabilistic radius $R_{A \times B}$ define by
$R_{A \times B}(x) = \sup_{t<x} \inf_{(a,b)\in A\times B} F_{a,b}(t)$ and $R_{A \times B}(+\infty) = 1$

Then the set A × B is said to be certainly bounded if

(i). $R_{A \times B}(x) = \sup_{x\in(0,+\infty)} \inf_{(a,b)\in A \times B} F_{a,b}(x) = 1$

And perhaps bounded if

(ii). $R_{A \times B}(x) = \sup_{x\in(0,+\infty)} \inf_{(a,b)\in A \times B} F_{a,b}(x) < 1$

More ever, A × B will said to be D-bounded if either (i) or (ii) holds. In shortly $\lim_{x\rightarrow\infty} R_{A \times B}(x) = 1$. Gh. Constantin [9] has introduced ε-continuous mapping between two PM spaces. In case of multi valued mapping, we have the next definition :

Definition 2.8: In GP 2-N space (L × M, F, τ) a map $f : L \times M \rightarrow L \times M$ is said to be ε-continuous and provided $R_{f(l,m)}(x) > 1 - \varepsilon$, for $p,q \in L \times M$ and $\varepsilon > 0$.

Lemma 2.1: Let (L × M, F, τ) be a GP 2-N space and $A \times B \subset L \times M$. If $f : A \times B \rightarrow A \times B$ be a ε-continuous, then for each $a,b \in A \times B$, $\varepsilon > 0$ we have $F_{f(a,b)}(x) > 1 - \varepsilon$.

Proof: Since f is a ε-continuous, for $a,b \in A \times B$, $\varepsilon > 0$. Therefore we have $R_{f(l,m)}(x) > 1 - \varepsilon$.

But for any $F_{a,b} \in F_{l,m}$, by theorem 2.1 (ii) we have $F_{a,b} \geq D_{A \times B}$. In similar way $F_{f(a,b)}(x) \geq R_{f(l,m)}(x)$.

Hence $F_{f(a,b)}(x) > 1 - \varepsilon$. This complete the proof.

3 MAIN RESULT

Let (L × M, F, τ) be GP 2-N space and $A \times B \subset L \times M$ and f be a multi valued self mapping like $f : A \times B \rightarrow A \times B$. The unique property $F_{f(a,b)-(a,b)}(x) = H_0$ known as fixed point of f, where $a \in A$ and $b \in B$. But for $x \in R > 0$, $F_{f(a,b)-(a,b)}(x) < 1$. Therefore $f(a,b) \neq (a,b)$, i.e. in A × B, f has no fixed point. In this case a natural question arises about the existence of any approximate fixed point. So that, $F_{f(a,b)-(a,b)}(x) \rightarrow H_0$ and $f(a,b) \cong (a,b)$. But if $F_{f(a,b)-(a,b)}(x) \rightarrow 1$, for $x \in R > 0$, we can observe that $f(a,b) \rightarrow (a,b)$. From this observation and assumption we define the artificial approximate fixed point and prove approximate fixed point theorem in Generalized probabilistic 2-normed space.

Definition 3.1: Let (L × M, F, τ) be a GP 2-N space and $A \times B \subset L \times M$. For $\varepsilon > 0$, $(a,b) \in A X\times B$ is an ε fixed point of $f : A \times B \rightarrow A \times B$ such that

$$\sup_{x<\varepsilon} F_{f(a,b)-(a,b)}(x) = 1$$

If the mapping f has at least one fixed point, we will say that f has approximate fixed point property. Refer [1] for similar notion in single valued mapping and approximate fixed point theorem in same situation. According to famous Brouwer's fixed point theorem "Each continuous function from a compact and convex subset of $V = R^m$ into itself possesses at least one fixed point". Here we give an extension to above thought, where we use D-boundedness condition instead compactness condition for the subset $A \times B \subset L \times M$. This replacement denies the presence of fixed point and then we show that at least one approximate fixed point still exist.

Theorem 3.1: Let A × B be a nonempty D – bounded and convex subset of (L × M, F, τ), that is GP 2-N space and f : A ×

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B → A × B be a ε-continuous function. Then f has the approximate fixed point.

Proof: Since f is an ε-continuous function, then according to Lemma 2.1

$$\sup_{x\in(0,+\infty)} F_{f(a,b)}(x) = 1, \text{ for } (a, b) \in A \times B.$$

Let C × D ⊂ A × B be a compact and convex subset, defined by C × D = (1 - α) $\overline{A \times B}$ Here $\overline{A \times B}$ is a probabilistic closure of A × B. Again take a continuous function g: C × D → C × D by g(c, d) = (1 - α) f(c, d), ∀ (c,d) ∈ C × D. If we apply Brouwer's fixed point theorem, then there should be exist (c', d') ∈ C × D such that g(c',d') = (c', d'). Which implies g(c',d') = (1 - α) f(c',d') = (c',d'). Where $F_{(1-\alpha) f(c',d') - (c',d')} = H_0$ Since (1 - α) f(c',d') – (c',d') = f(c',d') - α f(c',d') – (c',d') f(c',d') – (c',d') = α f(c',d') + (c',d') - (c',d') = α f(c',d') + f(c',d') - α f(c',d') - (c',d') = α f(c',d') + (1 - α)f(c',d') - (c',d') By PN3, we can write $F_{f(c',d') - (c',d')} \geq \tau (F_{f(c',d') - \alpha f(c',d')\ (c',d')}, F_{\alpha f(c',d')})$

$$= \tau \left(H_0, F_{f(c',d')}\left(\frac{t}{|\alpha|}\right)\right), \text{ for } t \in R > 0, \alpha \in R - \{0\}$$

$$\geq F_{f(c',d')}\left(\frac{t}{|\alpha|}\right).$$

By applying the sup condition on both sides of above inequality :

$$\sup_{0<t<\varepsilon} F_{f(c',d') - (c',d')}(t) \geq \sup_{0<t<\varepsilon} F_{f(c',d')}\left(\frac{t}{|\alpha|}\right)$$

Because (c', d') ∈ C × D ⊂ A × B, therefore from the definition 3.1 we have $\sup_{0<t<\varepsilon} F_{f(c',d')}\left(\frac{t}{|\alpha|}\right) = 1$ Then $\sup_{0<t<\varepsilon} F_{f(c',d') - (c',d')}(t) = 1$ Hence (c',d') is an approximate fixed point of f.

4 Conclusion

This research work declare the existence of approximate fixed point in Generalized Probabilistic 2-Normed Spaces.

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