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Deriving A Formula In Solving Fibonacci–Like Square Sequences

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Abstract: The coupled Fibonacci Sequences are first Introduced by K.T.Atanassov in 1985.Sequences have been fascinating topic for mathematics for centuries. Inclusion of missing terms in arithmetic, harmonic and geometric sequence has been formulated for a long time. other sequences like Fibonacci and Lucas Sequences could be solved using the Binet`s formula. In this paper, Derivation of formula in solving Fibonacci-like Square Sequence ,a derivative of Fibonacci Sequence, will be shown by finding important patterns from basic formula integrating with Binet`s Formula. Finding of missing terms in Fibonacci-Like square sequence will be answered easily using this formula

$$x_1 = \left(\frac{b^2 - \left[\frac{\phi^n - (1-\phi)^n}{\sqrt{5}} \right] a^2}{\frac{\phi^{n+1} - (1-\phi)^{n+1}}{\sqrt{5}}} \right), n \geq 1.$$

Key words: Fibonacci Sequence,Fibonacci like Sequence,Missing terms,Binet`sformula

Mathematics Subject Classification: 11B39, 11B37

Introduction

Many sequences have been studied for many years now.Arithmetic,Geometric,Harmonic,Fibonacci, and Lucas sequence have been very well defined in mathematical journals.On the other hand,Fibonacci-Like sequence received little attention from mathematicians since Fibonacci sequence attractes them more. Fibonacci sequence(1,1,2,3,5,8,13,21…) is a succession of rs that are obtained through adding the two preceding numbers.A derivative of this sequence is called Fibonacci-like sequence.This Sequence has properties like of Fibonacci except that it can start with any two numbers and follows the same patterns of adding them to the next 12,32,44,76… and so forth. Lucas and Tribonacci sequences are examples of this sequence.Generally,Fibonacci-Like sequence can be expressed as

$$S_n = S_{n-1} + S_{n-2}; n \geq 2 \qquad \text{…(1.1)}$$

And Fibonacci –like Square sequene define as

$$S_n^2 = S_{n-1}^2 + S_{n-2}^2; n \geq 2 \qquad \text{….(1.2)}$$

Where [x] is defined as the greatest integer less than of equal to x.Various properties of Fibonacci –Like Sequence have been presented in papers of B.Singh,et.[1] and other derivatives like that of composite number has been thoroughly explained in works of Nicol[2]. Inclusion of missing terms in arithmetic,harmonic,geometric sequence is well-solved in any high school and college mathematics textbooks.Also ,finding the nth terms of Fibonacci sequence can be solved using the Binet`s formula

Given by $F_n = \frac{\phi - (1-\phi)^n}{\sqrt{5}}$ … (1.3)

Where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803399....$ also known as the golden ratio.However,to the best of author`s knowledge, there are no scholarly works or studies describing a formula on inclusion of missing terms in a Fibonacci-like sequence e.g. insert five consecutive terms between 2 and 50 to make it a Fibonacci-Like sequence. P. Howell [3] presented a proof for finding nth term of Fibonacci sequence using vectors and eigenvalues but did not account Fibonacci-Like sequence.M.Agnes,et.al.[4] provided a formula for inclusion of missing terms in Fibonacci –like sequence but only for three consecutive missing terms. This paper will present how to derive a formula to solve a Fibonacci –like Square sequence from Binet`s formula regardless of the number of consecutive missing terms.

2. DERIVATION

Before moving to the general formula ,it is imperative to observe the specific formula from the basic problem.Basic formula will be studied and from this,a general will be deduced.in this derivation ,a^2 stands for the first term given and b^2 stands for last term given in any Fibonacci-Like square sequence.

2.1 One Missing Terms

Consider the Fibonacci –Like Square sequence a^2,_, b^2 where in there is one term missing denoted by x_1^2 ,we could

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solve the missing term because the next number in Fibonacci sequence is found by adding up the two numbers before it. In mathematical statement, $a^2 + x_1^2 = b^2$ .Rearranging terms,

$$x_1^2 = b^2 - a^2 \qquad \text{…(2.1.1)}$$

This basic formula will be used in finding will be used in finding the other formulas

2.2 Two Consecutive Missing Terms

Examining the Fibonacci-Like Square Sequence where a^2, x_1^2, x_2^2, b^2

Where x_1^2, is the first missing term and x_2^2,is the second missing terms,we could solve the formula. from (2.1.1),

$$x_1^2 = x_2^2 - a^2 \qquad \text{….(2..2.1)}$$

$$x_2^2 = b^2 - x_1^2 \qquad \text{….(2.2.2)}$$

Substituting (2.2.1) in (2.2.2) to find x_2^2,

$$x_2^2 = b^2 - (x_2^2 - a^2)$$

$$x_2^2 = \frac{b^2 + a^2}{2} \qquad \text{…(2.2.3)}$$

Substituting above in(2.2.1),we obtain

$$x_1^2 = \frac{b^2 + a^2}{2} - a^2$$

$$x_1^2 = \frac{b^2 - a^2}{2} \qquad \text{…..(2.2.4)}$$

2.3 Three Consecutive Missing Terms

The number of consecutive missing terms increases,finding the solution is getting complicated.Equation(2.1.1),(2.2.3), and(2.2.4) will be used to find the solution for this type of problem. Assuming a Fibonacci-Like Square sequence $a^2, x_1^2, x_2^2, x_3^2, b^2$ from (2.1.1),(2.2.3), and(2.2.4)

$$x_1^2 = x_2^2 - a^2 \qquad \text{…..(2.3.1)}$$

$$x_2^2 = x_3^2 - x_1^2$$

$$x_2^2 = x_3^2 - x_2^2 + a^2$$

$$x_2^2 = \frac{x_3^2 + a^2}{2} \qquad \text{…. (2.3.2)}$$

$$x_3^2 = b^2 - x_2^2 \qquad \text{….(2.3.3)}$$

Substituting (2.3.2) in (2.3.3) to find x_3^2,

$$x_3^2 = b^2 - \left(\frac{x_3^2 + a^2}{2}\right)$$

$$x_3^2 = \frac{2b^2 - a^2}{3} \qquad \text{…… (2.3.4)}$$

Rearranging terms in (2.3.3).we could solve x_2^2 as

$$x_2^2 = b^2 - x_3^2 = b^2 - \frac{2b^2 - a^2}{3} = \frac{b^2 + a^2}{3} \qquad \text{..;…..(2.3.5)}$$

Substituting above in (2.3.1) to find x_1^2

$$x_1^2 = x_2^2 - a^2 = \frac{b^2 + a^2}{3} - a^2 = \frac{b^2 - 2a^2}{3} \qquad \text{….. (2.3.6)}$$

2.4 Four Consecutive Missing Terms

We could easily find the formula for missing terms in Fibonacci-Like Square Sequence. $a^2, x_1^2, x_2^2, x_3^2, x_4^2, b^2$ using the same approach above Here are the solved formulas

$$x_1^2 = \frac{b^2 - 3a^2}{5} \qquad \text{…(2.4.1)}$$

$$x_2^2 = \frac{b^2 + 2a^2}{5} \qquad \text{…. (2.4.2)}$$

$$x_3^2 = \frac{2b^2 - a^2}{5} \qquad \text{….(2.4.3)}$$

$$x_4^2 = \frac{3b^2 + a^2}{5} \qquad \text{….(2.4.4)}$$

3. THE GENERAL FORMULA

TO FIND A GENERAL FORMULA FOR ALL n missing terms, a pattern must be recognized. all formula (2.1.1),(2.2.3),(2.2.4),(2.3.5),(2.3.6),(2.4.1)and (2.4.4)are

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tabulated in Table1 to find a recognizable pattern easily. Seemingly, all the missing terms in a Fibonacci-Like square sequence could be solved given the first missing terms x_1^2 since it is a succession of numbers. Those are obtained through adding the two preceding numbers. If a general formula will be obtained in x_1^2, the other missing terms will be calculated easily. A clear observation could be seen in Table 1 for x_1^2 that the numerical coefficient of a in numerator and the denominator of the formulas were following the Fibonacci Sequence as shown by table 2.This can be illustrated as

$$x_1^2 = \frac{b^2 - F_n a^2}{F_{n+1}} \quad \text{….(3.1)}$$

Note that the formula for finding the nth of Fibonacci sequence(also known as Binet`s formula)is

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} \quad \text{….(3.2)}$$

Table 1. formula for different missing terms in Fibonacci –Like Square sequence .

No.of missing term	Formula x_1^2	x_2^2	x_3^2	x_4^2
1	$b^2 - a^2$	……..	……………	………..
2	$b^2 - a^2 / 2$	$b^2 + a^2 / 2$	……	…….
3	$b^2 - 2a^2 / 3$	$b^2 + a^2 / 3$	$2b^2 - a^2 / 3$	
4	$b^2 - 3a^2 / 5$	$b^2 + 2a^2 / 5$	$2b^2 - a^2 / 5$	$3b^2 + a^2 / 5$

Table 2. Relationship of Number of Missing Terms wit

No.of Missing Term	Coefficient of a in Numerator	Coefficient of Denominator
1	1	1
2	1	2
3	2	3
4	3	5
.	.	.
.	.	.
n	$\frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$	$\frac{\phi^{n+1} - (1-\phi)^{n+1}}{\sqrt{5}}$ …

Numerator and Denominator of Formulas Using the Observation above,the general formula for x_1^2 is

$$x_1^2 = \frac{b^2 - \left[\frac{\phi^n - (1-\phi)^n}{\sqrt{5}}\right] a^2}{\frac{\phi^{n+1} - (1-\phi)^{n+1}}{\sqrt{5}}} \quad (3.3)$$

Where x_1^2 is the first missing term in Fibonacci-Like Square sequence, a^2 is the first term given ,$b^2$is the last term given,n is the number of missing terms and ϕ is known as the golden ration equal to 1.61803399… To fully understand the formua,we answer the presented problem by inserting five terms between 2^2 and 50^2 to make it a Fibonacci –Like Square sequence.In mathematical statement,the sequence is $2^2, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, 50^2, ...$ This problem will be solved by finding x_1^2 using (3.3) Given $a^2=2^2$, n=5, $b^2=50^2$

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$$x_1^2 = \frac{b^2 - \left[\frac{\phi^n - (1-\phi)^n}{\sqrt{5}}\right] a^2}{\frac{\phi^{n+1} - (1-\phi)^{n+1}}{\sqrt{5}}} = 5^2$$

Now that we have solved for x_1^2,we could easily find next terms Therefore the Fibonacci-Like Square sequence is 4,25,29,1466,2500…

4. CONCLUSION

$$\alpha_{n+7} = \beta_{n+6}.\beta_{n+5} = (\gamma_{n+5}\gamma_{n+4}).(\gamma_{n+4}\gamma_{n+3}) = \gamma_{n+5}.(\gamma_{n+4}^2).\gamma_{n+3}$$

(By Eighth Scheme)

$$= (\alpha_{n+4}.\alpha_{n+3})(\alpha_{n+3}.\alpha_{n+2})^2.(\alpha_{n+2}.\alpha_{n+1})$$

(By Eighth Scheme)

$$= \alpha_{n+4}.(\alpha_{n+3}\alpha_{n+2}).(\alpha_{n+3}\alpha_{n+2})^2.\alpha_{n+1}$$

(By Eighth Scheme)

$$= (\beta_{n+3}.\beta_{n+2}).\gamma_{n+4}^3.\alpha_{n+1}$$

(By Eighth Scheme)

$$= \beta_{n+3}.(\beta_{n+2}\alpha_{n+1}).\gamma_{n+4}^3.$$

(By Eighth Scheme)

$$= \beta_{n+3}.(\beta_{n+2}\beta_{n+1}).\gamma_{n+4}^3.$$

(By Induction hypo.)

$$= \beta_{n+3}.\alpha_{n+3}.\gamma_{n+4}^3.$$

(By Eighth Scheme)

$$\alpha_{n+7} = \alpha_{n+3}\beta_{n+3}\gamma_{n+4}^3$$

Hence result is true for all integers $n \geq 0$.Similar proofs can be given for remaining parts (b) and (c).

Theorem (3.2) For every integer $n \geq 0$

$$\alpha_n\beta_n\gamma_n = (\alpha_0\beta_0\gamma_0)^{F_{n-1}}.(\alpha_1\beta_1\gamma_1)^{F_n}.$$

Proof. To prove this, we shall use induction method. If n=1 the result is obviously true, since

$$\alpha_1\beta_1\gamma_1 = (\alpha_0\beta_0\gamma_0)^{F_{1-1}}.(\alpha_1\beta_1\gamma_1)^{F_1} = \alpha_1\beta_1\gamma_1.$$

Now suppose that the result is true for some integer $n \geq 2$ .Then

$$\alpha_{n+1}\beta_{n+1}\gamma_{n+1} = (\beta_n\beta_{n-1})(\gamma_n\gamma_{n-1})(\alpha_n\alpha_{n-1})$$

(By Eighth Scheme)

$$= (\alpha_{n-1}\beta_{n-1}\gamma_{n-1}).(\alpha_n\beta_n\gamma_n)$$

$$= (\alpha_0\beta_0\gamma_0)^{F_{n-2}}.(\alpha_1\beta_1\gamma_1)^{F_{n-1}}.(\alpha_0\beta_0\gamma_0)^{F_{n-1}}(\alpha_1\beta_1\gamma_1)^{F_n}$$

(By Induction hypo.)

$$= (\alpha_0\beta_0\gamma_0)^{F_{n-2}+F_{n-1}}.(\alpha_1\beta_1\gamma_1)^{F_n+F_{n-1}}$$

$$= (\alpha_0\beta_0\gamma_0)^{F_n}.(\alpha_1\beta_1\gamma_1)^{F_{n+1}}$$

Hence by induction method result is true for all $n \geq 1$.

4. Conclusion

Much work has been done on multiplicative coupled and triple Fibonacci sequences. In this paper, we have described some results of multiplicative triple Fibonacci sequence of second order under one specific scheme.

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